Is rentier capitalism that bad? Rent, efficiency and inequality dynamics

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Abstract. The current economic context shows a tendency to inequality and rather weak growth. Rent-seeking behavior is often blamed for that. The purpose of this paper is to analyze the consequences, on the accumulation trajectory, of the existence of a rent levied by the rich on the poor. The model is inspired by the articles Stiglitz (1969), Schilcht (1975) and Bourguignon (1981). In particular, convex saving is used. We seek to see to what extent the introduction of a rent may call into question the Pareto-superiority of inequality proved by Bourguignon (1981) or alter the risk of decline highlighted in Mabrouk (2016). Within the limits of the assumptions of the model and of the numerical simulations carried out, we arrive at interesting and rather unexpected observations. Namely, a moderate rent levied by the rich on the poor may not only allow a Pareto-improvement of the economy and prevent the risk of decline, but also, it may unlock the economy from under-accumulation trap even if initial capital endowment is insufficient. The disadvantages of such a rent for the poor are felt only if the economy approaches or exceeds the golden rule where the net marginal productivity of capital is zero.

Keywords. Inequality dynamics, Neoclassical growth, Rent, Efficiency.

JEL. D99, E13, E21, E22, O41.

1. Introduction

The current economic context shows a tendency to an increase in the income of the rich to the detriment of the poor[1]. Jacobs (2016) and Stiglitz (2015b) suggest that this increase in high incomes stems from rents with no clear counterpart in terms of output, such as rents due to market power, cronyism, or position rents due to the possession of irreplaceable assets such as well-situated buildings[2].

This situation is not in line with the neoclassical theory of income distribution according to marginal productivities that predicts that every factor earns a competitive income according to what it adds to domestic production. Would deviation from that theory have a negative impact on growth and economic efficiency? Although in the public debate the answer to this question tends to be positive, it is useful to look at it in more detail at the theoretical level[3].

The purpose of this paper is to analyze, within the framework of a simple neoclassical model, the consequences of the existence of a rent levied by the rich class on the competitive income of the poor class as set by the neoclassical theory of income distribution according to marginal productivities. This is done in a demonetized context, without uncertainty nor technical change, and taking into account the difference in saving behavior according to the level of income. The model is inspired by the articles Stiglitz (1969), Schilcht (1975) and Bourguignon (1981). The economy has two production factors: capital and labor, a production

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* The author expresses his thanks to Professor François Bourguignon for his comment

function with constant returns to scale, and an individual marginal propensity to save increasing with income. Individuals are assumed to be similar in all respects except for their membership in a given social class. This differentiates them only by their initial capital endowments and the rent received or paid.

Ignoring differences between individuals in terms of skills, saving behaviors and random events that could differentiate them, aims to focus on the impersonal aspect of inequalities dynamics. In this context, it appears that the assumption of a marginal propensity to save increasing with income (i.e. a convex saving function) is crucial for the emergence of distinct and stable social classes. Indeed, Stiglitz (1969) showed that a linear saving function leads to the convergence of classes. Even when considering a pseudo-convex saving function, where the marginal propensity to save passes discontinuously from 0 to a constant positive value when income increases, Stiglitz (2015a) shows that the only stable configuration remains a single social class. By extending the work of Stiglitz (1969) to the case of convex savings, Schilcht (1975) showed that one can get two stable classes. Bourguignon (1981) then showed that the equilibrium with two classes Pareto-dominates the egalitarian equilibrium.

Unlike Stiglitz (2015a) which focuses on inequality in itself and its causes, it should be noted that the present work is in the spirit of Bourguignon (1981), where the main concern is efficiency rather than inequality, and where egalitarian equilibrium is a poverty-trap from which one must escape. In this context, one seeks to see to what extent the introduction of a rent levied by the rich class on the income of the poor class may call into question the Pareto-superiority of the unequal configuration proved by Bourguignon (1981). We also want to see to what extent the introduction of such a rent alters the risk of decline highlighted in Mabrouk (2016).

After introducing the model and the assumptions in section 2, sections 3, 4 and 5 attempt to prepare the mathematical groundwork of the general model in order to show how rent modifies the curves that govern equilibrium under the conditions imposed in section 2. From section 6 on, since general calculations lack exploitable explicit formulas, we take a numerical example to follow the evolution of equilibria according to rent levels. This makes it possible to arrive at interesting, rather unexpected observations on the way in which rent influences the economic trajectory and the type of equilibrium. It should be noted that, although the parameters of the simulations are chosen at reasonable levels, these simulations do not pretend to have an empirical value.

Sections 7 and 8 study the equilibrium response to the variation of two essential parameters: the proportion of rich and the social propensity to save.

Charts are often used to base arguments. Charts without numerical values represent only the shapes of the curves and are drawn by hand. Those with numerical values are computed and plotted by computer.

2. Model and assumptions

The same assumption and notations as Mabrouk (2016) are used, except some specified below.

Individual savings are assumed to depend on income according to the function $S(y)$ where y is the income of the individual concerned. S is convex, increasing, twice differentiable on $]0, +\infty[$ and checks $S(0) = 0, S'(0) > 0$ and $\lim_{y \to \infty} S'(y) = 1$. Denote T the inverse function of S. We have $T' > 1, T'' < 0$ and $\lim_{x \to \infty} T'(x) = 1$. The per capita production function is $f(k)$ where k is the average capital per capita. f is increasing, concave, twice differentiable on $]0, +\infty[$ and checks $f(0) = 0$. The capital undergoes depreciation at a rate δ per unit of time and capital. k^* is the per capita capital of the golden-rule defined by $f'(k^*) = \delta$.

The society is composed of two classes: the poor, in proportion a_1 and the rich in proportion $a_2 = 1 - a_1$. We assume $a_2 < a_1$.

The following two conditions guarantee that we do not deviate too much from the case where the saving function is linear and where there exists a unique stable egalitarian equilibrium with non-zero production:

Condition 1 : $f'(0) > \delta T'(0)$

Condition 2 : There is a unique $\hat{k}$ such that $f'(\hat{k}) - \delta T'(\delta \hat{k}) = 0$

These conditions reduce the generality of this paper, but they allow to lighten the analysis while giving an idea of what can happen when the saving function is convex.

Proposition 3 shows that conditions 1 and 2 imply that the equation $f(k) - \delta T(\delta k) = 0$ has a unique solution $k_0 > 0$. This value is in fact the capital of the egalitarian equilibrium of the economy under consideration.

Like in Bourguignon (1981), assume that:

$$k_0 < k^* \tag{1}$$

The economic interpretation of assumption (1) is that the poor class does not generate enough savings to achieve maximum efficiency of the economy.

Instead of the usual neoclassical assumption that labor and capital are paid according to their respective marginal productivities, it is assumed that the wealthy class gains a rent μ in addition to its competitive income. The rent μ is levied by the rich class on the competitive income of the poor class.

By normalizing the size of the population to 1, per capita income in the rich class is:

$$f(k) - kf'(k) + c_2 f'(k) + \frac{\mu}{a_2}$$

Per capita income in the poor class is:

$$f(k) - kf'(k) + c_1 f'(k) - \frac{\mu}{a_1}$$

where c_1, c_2 are respectively per capita capital in the poor class and per capita capital in the rich class.

The dynamics of the economy are then characterized by the following differential system:

$$\dot{c}_1 = S\left[f(k) + (c_1 - k)f'(k) - \frac{\mu}{a_1}\right] - \delta c_1$$
$$\dot{c}_2 = S\left[f(k) + (c_2 - k)f'(k) - \frac{\mu}{a_2}\right] - \delta c_2$$
$$k = a_1 c_1 + a_2 c_2$$

By using T the inverse function of S, the equilibrium must satisfy the following system:

$$S\left[f(k) + (c_1 - k)f'(k) - \frac{\mu}{a_1}\right] = T(\delta c_1)$$
$$S\left[f(k) + (c_2 - k)f'(k) - \frac{\mu}{a_2}\right] = T(\delta c_2) \tag{2}$$
$$k = a_1 c_1 + a_2 c_2$$

Denote (E_1) and (E_2) the locus of the points in the space (k, c) defined respectively by the first and second equations of the system (2).

In the following, the curves (E_1) and (E_2) are constructed with the help of

graphic arguments.

3. The relationship between k, c_1 and c_2 at equilibrium

3.1. Plotting the curve (E_2)

By deriving the two equations of (E_1) and (E_2) with respect to k, we obtain an expression which gives the derivative of c with respect to k on (E_1) or (E_2):

$$\frac{dc}{dk}[f'(k) - \delta T'(\delta c)] = f''(k)(k - c) \qquad (3)$$

Denote (C) the locus of the points in the plane (k, c) checking:

$$\delta T'(\delta c) - f'(k) = 0$$

As explained in Bourguignon (1981), (C) is increasing, lies in the half-plane $(k < k^*)$ and admits the straight line $(k = k^*)$ as a vertical asymptote.

Proposition 3: There is a unique $k_0 > 0$ such that $f(k_0) - \delta T(\delta k_0) = 0$ and we have $\hat{k} < k_0$ and $f'(k_0) - \delta T'(\delta k_0) < 0$.

Proof: Define the function $\psi(k) = f(k) - T(\delta k)$. We have $\psi(0) = 0$ and $\psi'(k) = f'(k) - \delta T'(\delta k)$. By condition 1, $\psi'(0) > 0$. Moreover, since $f' = (k^*) = \delta$, there is $\delta' < \delta$ such that for k sufficiently large, we have $f'(k) < \delta'$. Thus, when tends towards $+\infty$ we have $\psi'(k) < \delta' - \delta T(\delta k) \to \delta' - \delta < 0$. Taking account of conditions 1 and 2 and since ψ' is continuous, we deduce that ψ' is positive on $[0, \hat{k}[$, zero at $\hat{k}$ and negative on $]\hat{k}, +\infty[$. Thus ψ' is increasing on $[0, \hat{k}[$ and decreasing on $]\hat{k}, +\infty[$. The properties concerning k_0 arise therefrom. **QED**

As stated above, k_0 is the equilibrium reached with a single social class, i.e. the egalitarian equilibrium. By virtue of the inequality $f'(k_0) - \delta T'(\delta k_0) < 0$, the egalitarian equilibrium k_0 is stable.

It follows that the solution $\underline{k_2}$ of the equation $\psi(k) = -\frac{\mu}{a_2}$ (for $\mu > 0$) is unique and satisfies $k_0 < \underline{k_2}$ and $f'(\underline{k_2}) - \delta T'(\delta \underline{k_2}) < 0$.
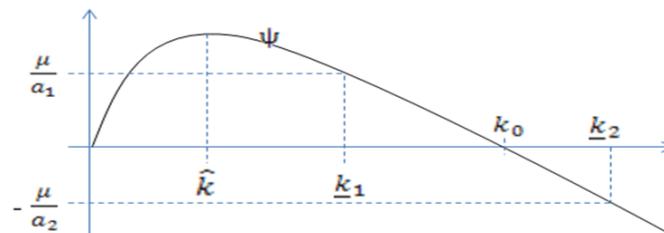
Figure 1.

The shape of ψ (figure 1) indicates that the expression $f'(k) - \delta T'(\delta c)$ evaluated on the line $(k = c)$ in the plane (k, c) is negative to the right of $\hat{k}$ and positive to the left (see figures 2 and 3). Therefore, in the plane (k, c), the point $(\underline{k_2}, \underline{k_2})$ lies in the area of the plane where $f'(k) - \delta T'(\delta c) < 0$. By (3), (E_2) crosses the line $(k = c)$ through $(\underline{k_2}, \underline{k_2})$ with a horizontal tangent. In the right neighborhood of $\underline{k_2}$, (E_2) is therefore below $(k = c)$. In the left neighborhood of $\underline{k_2}$, (E_2) lies above $(k = c)$. Since (E_2) crosses the line $(k = c)$ only in $\underline{k_2}$, the branch of (E_2) emanating from the right neighborhood of $(\underline{k_2}, \underline{k_2})$ always remains below $(k = c)$ and is increasing. The branch of (E_2) which emanates

from the left neighborhood of $(\underline{k_2}, \underline{k_2})$ always remains above $(k = c)$ and decreases until it encounters (C) as the case may be.

Proposition 4: In contrast to the curve (E) in (Mabrouk 2016), the introduction of μ causes two cases to occur: (E_2) intersects the vertical $(k = k^*)$ once or does not intersect it.

Proof: Consider the expression $E = T(\delta c) - \delta c - \left[f(k^*) - \delta k^* + \frac{\mu}{a_2} \right]$. The value of E at $c = 0$ is negative. The derivative of E with respect to c is: $E = \delta \left(T^{'(\delta c)} - 1 \right) > 0$. E is then increasing as a function of c. Its maximum is $\max_c E = \lim_{c \to +\infty} [T(\delta c) - \delta c] - \left[f(k^*) - \delta k^* + \frac{\mu}{a_2} \right]$. Denote $\mu_0 = a_2 (\lim_{c \to +\infty} [T(\delta c) - \delta c] - [f(k^*) - \delta k^*])$. Assumption (1) implies $\psi(k^*) < 0$, i.e. $f(k^*) - \delta k^* < 0$. By evaluating the expression $[T(\delta c) - \delta c] - [f(k^*) - \delta k^*]$ at $c = k^*$, we get $T(\delta k^*) - f(k^*)$. We thus have $\frac{\mu_0}{a_2} = \max_c [T(\delta c) - \delta c] - [f(k^*) - \delta k^*] \geq T(\delta k^*) - f(k^*) > 0$. Thus $\mu_0 > 0$.

It follows that if $\mu < \mu_0$, then the expression $[T(\delta c) - \delta c] - \left[f(k^*) - \delta k^* + \frac{\mu}{a_2} \right]$ takes the value 0 for some c^* in $]0, +\infty[$. Thus the curve (E_2) intersects the vertical $(k = k^*)$ at (k^*, c^*). If $\mu \geq \mu_0$, then there is no c^* such that $(k^*, c^*) \in (E_2)$ **QED.**

$$\text{Case 1: } \mu \geq \mu_0$$

Proposition 5: (E_2) is entirely to the right of the vertical $(k = k^*)$ and this vertical is an asymptote to (E_2).

Proof: For a given k , assume there exists $c \geq 0$ such that $f(k) + (c - k)f^{'}(k) + \frac{\mu}{a_2} = T(\delta c)$. We thus have $\frac{\mu}{a_2} = T(\delta c) - f(k) - (c - k)f^{'}(k) \geq \frac{\mu}{a_2} = \lim_{x \to +\infty} [T(\delta x) - \delta x] - [f(k^*) - \delta k^*]$. Hence, $T(\delta c) - cf^{'}(k) \geq \max_x [T(\delta x) - \delta x] - [(f(k^*) - k^* f^{'}(k^*)) - (f(k) - kf^{'}(k))]$. If $k \to k_+^*$, this inequality can be written $T(\delta c) - cf^{'}(k) \geq \max_x [T(\delta x) - \delta x] - \varepsilon$, where ε is as small as one wants. This shows that c tends to $+\infty$ since the maximum of $[T(\delta x) - \delta x]$ is reached when $x \to +\infty$. Therefore the vertical $(k = k^*)$ is an asymptote to (E_2).

If $k \to k_-^*$, then for all $x \geq 0$ we have
$$T(\delta c) - cf^{'}(k) \geq [T(\delta x) - \delta x] - [(f(k^*) - k^* f^{'}(k^*)) - (f(k) - kf^{'}(k))]$$
Take $\varepsilon^{'} = x - c$ positive and close to 0. Then take k as close as necessary to k_-^* so that the quantity $[(f(k^*) - k^* f^{'}(k^*)) - (f(k) - kf^{'}(k))]$ be negligible in comparison with $[T(\delta x) - T(\delta c)]$. This gives the inequality $\delta x - cf^{'}(k) \geq [T(\delta x) - T(\delta c)] \geq 0$. For $\varepsilon^{'}$ sufficiently close to 0_+, the latter inequality gives $\delta - f^{'}(k) \geq 0$, which is impossible for $k < k^*$. We deduce that the curve (E_2) does not pass in the left neighborhood of k^*. Therefore, the curve (E_2) does not pass in the area $[0, k^*]$ because, assuming the opposite and using (3), we would get step by step to the left neighborhood of k^* **QED.**

Remark 6: It is useful for the following to observe that since (E_2) does not intersect the area $[0, k^*]$ when $\mu \geq \mu_0$, for capital to equal k^* at equilibrium, it is necessary to have $\mu < \mu_0$.
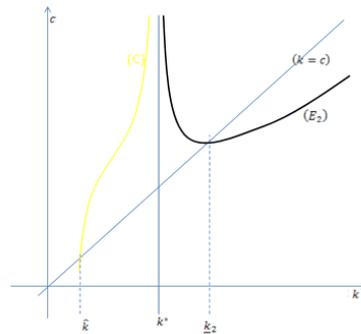
Figure 2.

Case 2: $\mu < \mu_0$

This case is similar to the case addressed in (Bourguignon 1981). The branch of (E_2) which emanates from the left neighborhood of $\underline{k_2}$ intersects (C) at a point denoted (k_{v2}, c_{v2}). According to (3), the tangent to (E_2) at point (k_{v2}, c_{v2}) is vertical. (E_2) becomes increasing as soon as it passes above (C) at (k_{v2}, c_{v2}).

When k increases from k_{v2}, this branch can not intersect again (C) because it should do so with a vertical slope, which is not possible since (C) does not have any vertical tangent. Therefore it remains above (C). Note that $\delta T^{'}(\delta c_{v2}) = f^{'}(k_{v2})$ implies $f^{'}(k_{v2}) > \delta$. So $k_{v2} < k^*$. When k tends to k^* from the left, the branch of (E_2) above $(k = c)$ admits a vertical asymptote like (C).
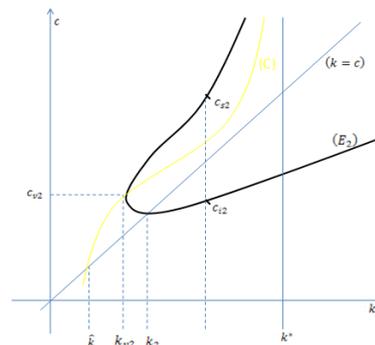
Figure 3.

We now give some properties that help to see the changes that take place when μ varies.

We have the following inequalities $\hat{k} < k_{v2} < \underline{k_2} < c_{v2}$ and $k_{v2} < k^*$.

Proposition 7: $\lim_{\mu \to \mu_0-} k_{v2} = k^*$

Proof: Since $\underline{k_2} = \psi^{-1}\left(-\frac{\mu}{a_2}\right)$, $\underline{k_2}$ varies continuously with respect to μ. We know that for $\mu = \mu_0$, we have $k_{v2} > k^*$. Therefore, when $\mu \to \mu_0 -$, we have $\lim_{\mu \to \mu_0-} k_{v2} > k^*$.

Therefore, when $\mu \to \mu_0 -$, (E_2) is decreasing between k_{v2} and k^*, so $c_{v2} > c^*$. It is now sufficient to see that $\lim_{\mu \to \mu_0-} c^* = +\infty$ to deduce that $\lim_{\mu \to \mu_0-} c_{v2} = +\infty$, and, being on the curve (C), to deduce that $\lim_{\mu \to \mu_0-} k_{v2} = k^*$. Indeed, $\mu \to \mu_0 -$ can be written as:

$$\frac{\mu}{a_2} = [T(\delta c^*) - \delta c^*] - [f(k^*) - \delta k^*] \geq \frac{\mu_0}{a_2}$$
$$= \lim_{c \to +\infty} [T(\delta c) - \delta c] - [f(k^*) - \delta k^*]$$

which entails $\lim_{\mu \to \mu_0-} c^* = +\infty$ **QED.**

Proposition 8: k_{v2} is increasing as a function of μ.

Proof: Differentiate $f(k_{v2}) + (c_{v2} - k_{v2})f'(k_{v2}) + \frac{\mu}{a_2} = T(\delta c_{v2})$ with respect to μ along the curve (C). We get: $k'_{v2} = \frac{1}{a_2(k_{v2} - c_{v2})f''(k_{v2})} > 0$ **QED.**

Proposition 9: $\underline{k_2}$ is increasing as a function of μ and $\lim_{\mu \to +\infty} \underline{k_2} = +\infty$

Proof: The function $f(k)$ has an asymptotic direction with a slope strictly less than δ and the function $T(\delta k)$ has an asymptotic direction with slope δ. Therefore $\lim_{k \to +\infty} \psi(k) = f(k) - T(\delta k) = -\infty$. Equation $\psi(\underline{k_2}) = -\frac{\mu}{a_2}$ implies $\lim_{\mu \to +\infty} \underline{k_2} = +\infty$. By differentiating the expression $\psi(\underline{k_2}) = -\frac{\mu}{a_2}$ with respect to μ, we get: $\psi'(\underline{k_2})\underline{k_2}'_\mu = -\frac{1}{a_2}$. But $\psi'(\underline{k_2}) < 0$. So $\underline{k_2}'_\mu > 0$ **QED.**

It is useful for the following to see the solutions of the second equation of (2) in another way. Denote by $X_2(c)$ the expression $f(k) + (c - k)f'(k) + \frac{\mu}{a_2}$, considering k as a parameter and c as a variable; and denote by $Y(c)$ the expression $T(\delta c)$. The function T is concave and its derivative satisfies $T' > 1$. Therefore the function Y is concave and its derivative satisfies $Y' > \delta$.

We are now in the plane (c, X_2). In the case $\mu < \mu_0$, X_2 and Y are tangent at the point c_{v2} for $k = k_{v2}$. If k increases, according to figure 3, we obtain two intersections c_{s2} and c_{i2} so long as the asymptotic slope of Y, which is δ, is less than the slope of X_2, which is $f'(k)$, i.e. as long as $k < k^*$. As soon as k exceeds k^*, the line X_2 flips as shown in figure 4. The point c_{s2} is rejected at infinity and the intersection becomes only c_{i2}.
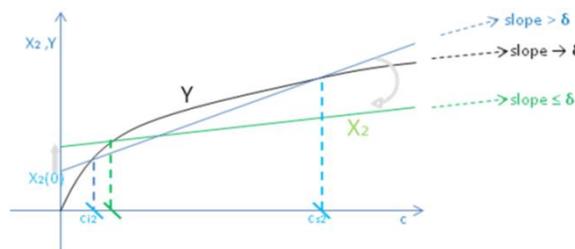
Figure 4.

If $\mu \geq \mu_0$ and if $k \leq k^*$, there is no intersection between Y and X_2. If $k > k^*$, the intersection is limited to a single point.

3.2. Plotting the curve (E_1)

Figure 1 shows that under the assumption:

$$\mu < \mu_1 = a_1\psi(\hat{k}) \tag{4}$$

equation $\psi(k) = \frac{\mu}{a_1}$ has two solutions, the largest of which, denoted $\underline{k_1}$, is greater than $\hat{k}$.

We shall limit ourselves to the cases where condition 4 is satisfied.[4]

We are interested only in the solution of the first equation of system (2) which is greater than $\hat{k}$. Indeed, (E_2) lies entirely on the right of $\hat{k}$ and therefore there can not be a pair (c_1, c_2) that verifies the first two equations of (2) if $k \leq \hat{k}$.

To the right of $\hat{k}$, the pair $(\underline{k_1}, \underline{k_1})$ is solution of the first equation of (2). The curve (E_1) is constructed in the plane (k,c) starting from the point $(\underline{k_1}, \underline{k_1})$ in the same way as (E_2).

Denote $X_1(c)$ the expression $f(k) + (c-k)f'(k) - \frac{\mu}{a_1}$. The representation of $X_1(c)$ is added to figure 4 by observing that the two straight lines $X_1(c)$ and $X_2(c)$ are parallel and that $X_1(0) < X_2(0)$.
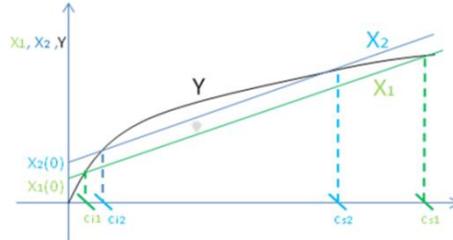
Figure 5.

Therefore, as long as $X_1(0) > 0$ and X_2 intersects Y at two points, X_1 intersects Y at two non-zero points c_{i1} and c_{s1} such that $c_{i1} < c_{i2}$ and $c_{s1} > c_{s2}$. In the plane (k, c), the upper branch of (E_1) will be above the upper branch of (E_2) and the lower branch of (E_1) will lie below the lower branch of (E_2). Condition $X_1(0) > 0$ amounts to $f(k) - kf'(k) - \frac{\mu}{a_1} > 0$. Denote by $\varphi(k) = f(k) - kf'(k)$. φ is increasing on $[0, +\infty[$ and $\varphi(0) = 0$ (because the concavity of f and $f(0) = 0$ gives $kf'(k) < f(k)$, hence $\lim_{k \to 0} kf'(k) = 0$).

Condition $X_1(0) > 0$ is equivalent to: $\varphi(k) > \frac{\mu}{a_1}$.

In order to confirm the construction of the curve (E_1), carried out similarly to (E_2), the following two properties are proved:

Proposition 10: Condition $X_1(0) > 0$ is satisfied as long as $k > \hat{k}$.

Proof: For $k > \hat{k}$ we have $\psi'(k) < 0$. Thus $f'(k) < \delta T'(\delta k)$. Moreover, by concavity of T and $T(0) = 0$, the function $T(\delta k) - \delta k T'(\delta k)$ is increasing in k and is zero for $k = 0$ .Thus $T(\delta k) - \delta k T'(\delta k) > 0$ for $k > 0$. To sum up: $kf'(k) < \delta k T'(\delta k) < T(\delta k)$. This gives $\psi(k) = f(k) - T(\delta k) < f(k) - kf'(k) = \varphi(k)$ for $k > \hat{k}$. For $k \in]\hat{k}, \underline{k_1}]$, we then get $\varphi(k) > \psi(k) \geq \psi(\underline{k_1}) = \frac{\mu}{a_1}$. And for $k > \underline{k_1}$, we get $\varphi(k) > \varphi(\underline{k_1}) \geq \psi(\underline{k_1}) = \frac{\mu}{a_1}$. We have proven that if $k > \hat{k}$ then $\varphi(k) > \frac{\mu}{a_1}$ **QED.**

Proposition 11: The first equation of (2) does not admit a solution at $k = \hat{k}$ (a fortiori the second equation - see figure 5).

Proof: Suppose there is c_1 such that $f(\hat{k}) + (c_1 - \hat{k})f'(\hat{k}) - \frac{\mu}{a_1} = T(\delta c_1)$.

Subtract $T(\delta \hat{k})$ from the two sides of the latter equation. It gives:

$$\left(f(\hat{k}) - T(\delta \hat{k}) - \frac{\mu}{a_1} \right) + (c_1 - \hat{k})f'(\hat{k}) = T(\delta c_1) - T(\delta \hat{k})$$

But $\left(f(\hat{k}) - T(\delta\hat{k}) - \frac{\mu}{a_1}\right) > 0.$ Thus $(c_1 - \hat{k})f'(\hat{k}) < T(\delta c_1) -$ $T(\delta\hat{k})$. Replace $f'(\hat{k})$ by $\delta T'(\delta\hat{k})$. It gives: $(c_1 - \hat{k})\delta T'(\delta\hat{k}) < T(\delta c_1) -$ $T(\delta\hat{k})$. The latter inequality is impossible since $T(\delta k)$ is concave **QED.**

Thus, by decreasing k towards $\hat{k}$ from $\underline{k}_1$, the intersection between the line X_1 and Y passes from 2 points to 0 point, knowing that the abscissas of the points of intersection, when they exist, are in $]0, +\infty[$. Thus X_1 "detaches" from Y before k reaches $\hat{k}$. By continuity, this necessarily occurs when X_1 and Y become tangent for some value of k denoted k_{v1}.

We thus have $\hat{k} < k_{v1} < \underline{k}_1 < c_{v1}$.

Figure 1 shows that $\lim_{\mu \to \mu_1} \underline{k}_1 = \hat{k}$. We deduce $\lim_{\mu \to \mu_1} k_{v1} = \hat{k}$. Therefore, c_{v1} being the image of k_{v1} on the curve (C), we also have $\lim_{\mu \to \mu_1} c_{v1} = \hat{k}$.

Since ψ is decreasing on $[\hat{k}, +\infty[$ and $\psi(\underline{k}_1) = \frac{\mu}{a_1} > 0 = \psi(k_0)$, we have $\underline{k}_1 < k_0 < k^*$. This allows to construct the curve (E_1) starting from the point $\underline{k}_1$ as we have done for (E_2) when $\mu < \mu_0$.

We easily establish the following formulas which show that k_{v1} and $\underline{k}_1$ are decreasing as functions of μ :

$$k'_{v1} = -\frac{1}{a_1(k_{v1} - c_{v1})f''(k_{v1})} < 0$$
$$\underline{k}_1 = \frac{1}{a_1\psi'(\underline{k}_1)} < 0$$

Figure 6 gives the shapes of the curves (E_1) and (E_2) for $\mu < \mu_0$ and $\mu \geq \mu_0$:
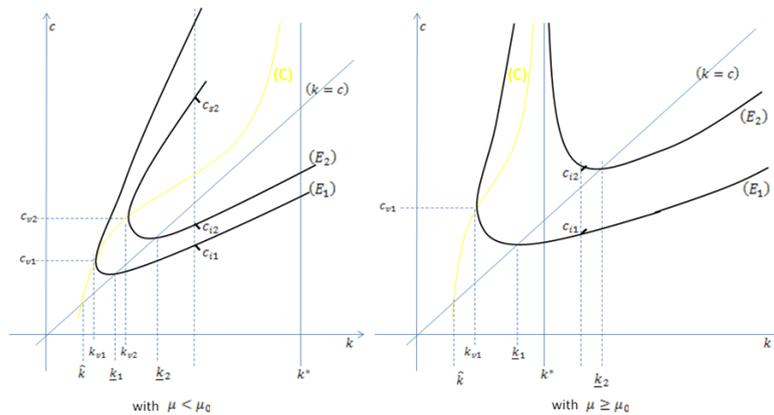
Figure 6.

4. As a first approach: the case where μ is close to 0

From now on we add the assumption: f 3 times differentiable on $]0, +\infty[$ and $f''' > 0$. This assumption is verified by the standard production functions. For μ sufficiently small, $k_{v2}(\mu)$ is close to $k_{v2}(0)$ and $\underline{k}_1(\mu)$ is close to $\underline{k}_1(0)$ Moreover, $k_{v2}(0) = k_{v1}(0) < \underline{k}_1(0)$. So we have $k_{v2}(\mu) < \underline{k}_1(\mu)$. The following shape is obtained:
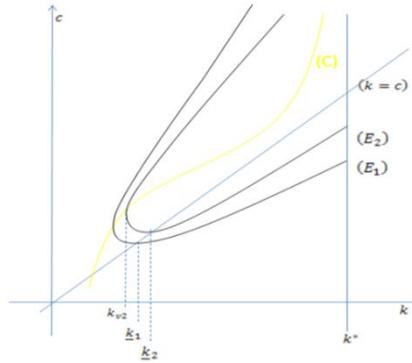
Figure 7.

For $k \in [\underline{k_1}, \underline{k_2}]$ define the function $A_i(k)$ by the equality $k = A_i(k)c_{i2}(k) + (1 - A_i(k))c_{i1}(k)$.

$A_i(k)$ is continuous. It is positive on $]\underline{k_1}, \underline{k_2}]$, zero at $\underline{k_1}$ and it takes the value 1 at $\underline{k_2}$. From now on, it is assumed that the system (2) is smooth enough for the functions $c_1(k)$ and $c_2(k)$ to be differentiable.

Proposition 12: $A_i(k)$ is increasing on $[\underline{k_1}, \underline{k_2}]$.

Proof: The assumption $f''' > 0$ is used here. The denominator of the expression of $A_i(k)$ is decreasing on $[\underline{k_1}, \underline{k_2}]$ since c_{i2} is decreasing and c_{i1} is increasing on this interval. Let us show that $k - c_{i1}(k)$ is increasing as a function of k. This is equivalent to showing that $1 - \frac{dc_{i1}}{dk} > 0$. Using equation (3), we get:

$$1 - \frac{dc_{i1}}{dk} = 1 - \frac{f''(k)(k - c_{i1})}{f'(k) - \delta T'(\delta c_{i1})}$$

We have to show that $\frac{f''(k)(k - c_{i1})}{f'(k) - \delta T'(\delta c_{i1})} < 1$. Observe that below the curve (c) the quantity $\delta T'(\delta c_{i1}) - f'(k)$ is positive. We thus have to show that $-f''(k)(k - c_{i1}) < \delta T'(\delta c_{i1}) - f'(k)$. Since $c_{i1} > \hat{k}$, we have $\psi'(c_{i1}) < 0$, thus $f'(c_{i1}) < \delta T'(\delta c_{i1})$. Therefore, we shall have attained our objective if we show that $-f''(k)(k - c_{i1}) < f'(c_{i1}) - f'(k)$. This last inequality follows from the assumption $f''' > 0$ which implies that f' is convex **QED**

The properties "$A_i(k)$ increasing on $[\underline{k_1}, \underline{k_2}]$ ", "$A_i(\underline{k_1}) = 0$" and "$A_i(\underline{k_2}) = 1$" show that for $a_2 \in [0,1]$ there exists a unique k_0' such that $A_i(k_0') = a_2$. The triplet $(k_0', c_{i1}(k_0'), c_{i2}(k_0'))$ is therefore a solution of system (2). If $\mu \to 0$, then $\underline{k_1} \to 0$ and $\underline{k_2} \to 0$. The system (2) can be linearized around 0 for μ close to 0. Denote:

$$k - k_0 = x$$
$$c_{i1} - k_0 = y$$
$$c_{i2} - k_0 = z$$

The first equation of system (2) becomes

$$f(k_0) + xf'(k_0) + (y - x)f'(k_0) - \frac{\mu}{a_1} \approx T'(\delta k_0)\delta y$$

Thus

$$y \approx \frac{\mu}{a_1 \psi'(k_0)}$$

Similarly, we establish the approximation

$$z \approx -\frac{\mu}{a_2 \psi'(k_0)}$$

and

$$x \approx 0$$

Since $\psi'(k_0) < 0$, we have $y < 0$ and $z > 0$. Average capital at equilibrium is almost equal to the egalitarian equilibrium capital k_0. But the poor class is worse off and the rich class is better off.

We are now interested in the possible equilibria on the lower branch of (E_1) and the upper branch of (E_2). These equilibria can be seen as the result of deformations following the introduction of a rent, of inegalitarian equilibria in the case without rent studied in Schilcht (1975), Bourguignon (1981) and Mabrouk (2016).

For $k \in [\underline{k}_1, k^*[$ define the function $A_s(k)$ by the equality $k = A_s(k)c_{s2}(k) + (1 - A_s(k))c_{i1}(k)$. In the same way as in [Mabrouk 2016], we see that $A_s(k)$ is zero in $\underline{k}_1$, positive on $]\underline{k}_1, k^*[$ and $\lim_{k \to k^*} A_s(k) = 0$. Consequently $A_s(k)$ admits a maximum on $]\underline{k}_1, k^*[$. This maximum is given by the resolution of the system of 6 unknowns $c_1, c_2, \frac{dc_1}{dk}, \frac{d2}{dk}, k$ and $\overline{A}$ the 6 equations given in Mabrouk (2016).

However, unlike Mabrouk (2016), $\overline{A}$ depends on a_1 and a_2 because the curves (E_1) and (E_2) depend on a_1 and a_2. The same kind of reasoning as in Bourguignon (1981) shows that equilibria on the lower branch of (E_1) and the upper branch of (E_2) occur in peers and that the equilibrium with the highest value of capital is stable. If μ is close enough to 0, this equilibrium is close to the stable Pareto dominant equilibrium of Bourguignon (1981). Hence, it is Pareto-superior to the egalitarian equilibrium 0. This does not fundamentally alter the conclusions obtained in the case without rent.

For $k > k_{v2}$, notice that $c_{s2}(k) > c_{i2}(k)$ implies $A_i(k) > A_s(k)$. Figure 8 and figure 9 show the possible shapes for the curves $A_i(k)$ and $A_s(k)$ when μ is small.
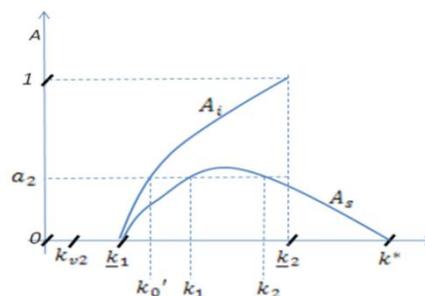
Figure 8. Pattern I

In figure 8, the horizontal $(A = a_2)$ intersects A_i and A_s. As in [Bourguignon 1981], one shows that k_0' is stable, k_1 is unstable, k_2 is stable and Pareto-dominant. We call k_0' the lower stable equilibrium because $c_2(k_0')$ is taken on

the lower branch of (E_2). We call k_1 the unstable equilibrium and k_2 the upper stable equilibrium because $c_2(k_2)$ is taken on the upper branch of (E_2). We are interested only in stable equilibria.

In figure 9, the horizontal $(A = a_2)$ intersects A_i only. There is only the lower stable equilibrium k_0'. Therefore, even if capital per capita is high at the outset, the economy will decline towards k_0'. The analysis is similar to the case where $a_2 > \overline{A}$ in Mabrouk (2016).
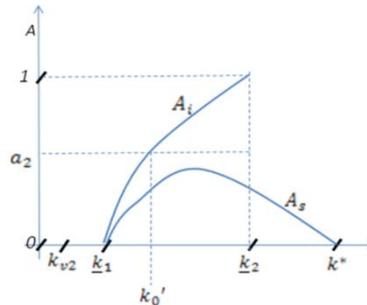
Figure 9. Pattern II

5. Case $k_{v2} > \underline{k_1}$

For $\mu \geq \mu_0$ let's agree to write $k_{v2} = k^*$ and $c_{v2} = +\infty$. The case $k_{v2} > \underline{k_1}$ can occur if μ increases sufficiently. Indeed, since $k_{v2}(0) < \underline{k_1}(0)$, $\lim_{\mu \to \mu_0-} k_{v2} = k^*$, $\underline{k_1} \leq k_0 < k^*$ and since k_{v2} is increasing and $\underline{k_1}$ is decreasing with respect to μ, there exists a unique μ_2 such that $k_{v2} = \underline{k_1}$. We have $\mu_2 < \mu_0$. For $\mu \in [0, \mu_2[$ we have $k_{v2} < \underline{k_1}$ and for $\mu > \mu_2$ we have $k_{v2} > \underline{k_1}$.

μ_2 is solution to the following system with the three unknowns k, c, μ and the three equations:

$$f(k) - T(\delta k) = \frac{\mu}{a_1}$$
$$f(k) + (c - k)f'(k) + \frac{\mu}{a_2} = T(\delta c)$$
$$f'(k) - \delta T'(\delta c) = 0$$

If $\mu > \mu_2$, the minimum value of $A_i(k)$ is no longer 0 since the minimum value of k is henceforth k_{v2}. Let $\underline{A}$ be this minimum value $\underline{A}$ is positive and we have:

$$\underline{A} = A_i(k_{v2}) = \frac{k_{v2} - c_{i1}(k_{v2})}{c_{v2} - c_{i1}(k_{v2})}$$

The domain of function A_s is now $[k_{v2}, k^*[$. Therefore, A_s no longer starts at the value 0 but at the value $\underline{A} = A_i(k_{v2})$.

We no longer have the assurance that A_s reaches a maximum inside $]k_{v2}, k^*[$ or that the derivative of A_s takes the value 0. However, we are certain that, in the plane (k, A) the horizontal $(A = a_2)$ intersects either A_i or A_s or both. Here are the possible patterns for the intersection of $(A = a_2)$ with A_i and A_s:
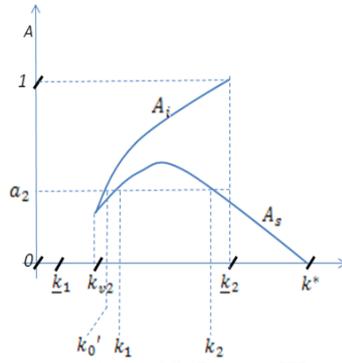
Figure 10. Pattern III

In pattern III, the analysis does not differ from that of pattern I.
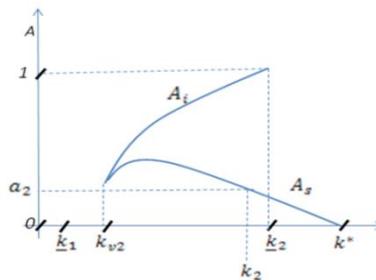
Figure 11: Pattern IV

In pattern IV, the lower stable equilibrium disappears, but not the upper stable equilibrium $\underline{k}$.
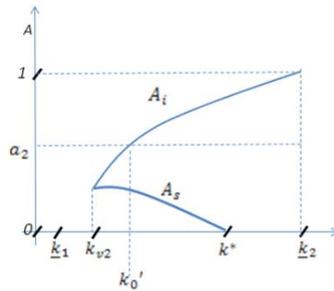
Figure 12. Pattern V

In pattern V, there is only equilibrium $\underline{k}$. The position of this equilibrium on A should not suggest that the value of $\underline{k}$ is small. It will be seen that $\underline{k}$ reaches high values for s sufficiently large.
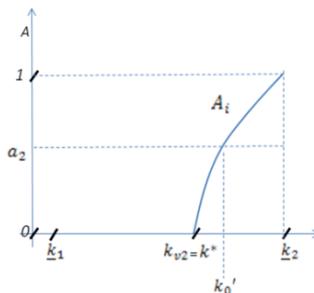
Figure 13. Pattern VI

Figure 13 represents A_i when $\mu \geq \mu_0$. The curve A_s disappears in this case because the upper branch of (E_2) no longer exists when $\mu \geq \mu_0$. The analysis of the equilibrium does not differ from that of pattern V.

Furthermore, $\underline{A}(\mu)$ is zero for $\mu \in [0, \mu_2]$ and positive for $\mu \in]\mu_2, \mu_0[$.

Functions f and T are supposed to be sufficiently smooth for the variables $k_{v2}, c_{v2}, c_{i1}(k_{v2})$ to be continuous with respect to μ. Consequently, $\underline{A}(\mu)$ is continuous with respect to μ over the interval $[0, \mu_0[$. It has been shown above that $\lim_{\mu \to \mu_0-} k_{v2} = k^*$ and $\lim_{\mu \to \mu_0-} c_{v2} = +\infty$. We deduce that $\lim_{\mu \to \mu_0-} \underline{A}(\mu) = 0$. For $\mu \geq \mu_0$ we agree to write $\underline{A}(\mu) = 0$. If $\max_\mu \underline{A}(\mu) > a_2$, we obtain the following figure:
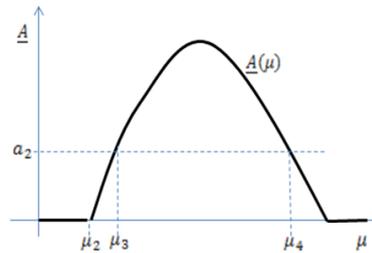
Figure 14.

with $\mu_3 = inf\{\mu : \underline{A}(\mu) = a_2\}$ and $\mu_4 = sup\{\mu : \underline{A}(\mu) = a_2\}$. If it is not the case, one moves directly from pattern II to pattern V and then VI. Changing the saving function may yield $\max \underline{A}(\mu) < a_2$. This is discussed in section 8.

If $\mu = \mu_3$ or $\mu = \mu_4$, we obtain an equilibrium which lies at the point of coordinates $(k_{v2}, minA_i)$ in the plane (k, A). Thus, in the plane (k, c), the corresponding point (k, c_2) is none other than (k_{v2}, c_{v2}) and lies on the curve (C).

Remark 13: The above entails $\mu_3 \leq \mu_4 < \mu_0$.

Proposition 14: The derivative with respect to μ of the net income of the poor at k_{v2} is zero for $\mu = \mu_i$, $i = 3$ or 4.

Proof: The point $(k, c_2) = (k_{v2}, c_{v2})$ satisfies the equation of (C) : $\delta T'(\delta c_2) - f'(k) = 0$. If we add this equation to the 3 equations of the system (2), we obtain 4 equations for the four unknowns k, c_1, c_2, μ_i. By combining the first two equations of (2), we get:

$$f(k) = a_1 T(\delta c_1) + a_2 T(\delta c_2) \tag{5}$$

For any μ, the solution (k, c_1, c_2) of the system (2) can be considered as a function $(k(\mu), c_1(\mu), c_2(\mu))$ of μ. Differentiate (5) with respect to μ. It gives:

$$f'(k)k_\mu{}' = \delta a_1 c'_{1\mu} T'(\delta c_1) + \delta a_2 c'_{2\mu} T'(\delta c_2)$$

Now take again $\mu = \mu_i$. Replace $f'(k)$ by its value given by the equation of (C). It gives:

$$\delta T'(\delta c_2)k'_\mu = \delta a_1 c'_{1\mu} T'(\delta c_1) + \delta a_2 c'_{2\mu} T'(\delta c_2)$$

Now replace k'_μ by $a_1 c'_{1\mu} + a_2 c'_{2\mu}$. It gives:

$$\delta T'(\delta c_2)\left(a_1 c'_{1\mu} + a_2 c'_{2\mu}\right) = \delta a_1 c'_{1\mu} T'(\delta c_1) + \delta a_2 c'_{2\mu} T'(\delta c_2)$$

After rearranging:

$$\delta a_1 c'_{1\mu} T'(\delta c_2) = \delta a_1 c'_{1\mu} T'(\delta c_1)$$

Thus $c'_{1\mu} T'(\delta c_2) = c'_{1\mu} T'(\delta c_1)$. Since $c_1 \neq c_2$, we have necessarily $c'_{1\mu} = 0$. The income of the poor is: $f(k) + (c_1 - k)f'(k) - \frac{\mu}{a_1} - \delta c_1$. The first equation of (2) allows us to write this income as: $T(\delta c_1) - \delta c_1$. The derivative of this expression with respect to μ is: $(T'(\delta c_1) - 1)\delta c'_{1\mu} = 0$ **QED.**

The economic interpretations of μ_3, μ_4 and proposition 14 will be developed in the following sections.

6. A numerical example
6.1. General data

We adopt the parameters used in Mabrouk (2016)[5] . The numerical values are only intended to highlight the economic phenomena that are being analyzed. They are chosen at levels supposed to be reasonable. But the question of conformity of these numerical values with the reality of a given country is not considered here not to clutter up this paper. The production function is chosen in such a way that it gives a gross income normalized to 1 with a capital coefficient of 2.5 (i.e. $f(2.5) \approx 1$). This makes it possible to interpret the values of the rent in terms of percentage of the gross income normalized to 1 considered as reference income. For example, $\mu = 1.5.10^{-2}$ is interpreted as a rent of 1.5% of the reference income.

We take $f(k) = \frac{3}{4} k^{0.3}$. The rate of capital depreciation is 3.7%. The saving function is constructed to meet the conditions of section 2 and realize savings rates ranging from 10% to 30% depending on income levels.

The formula chosen is:

$$S(y) = b + \frac{1}{2}(1 + c)(y - a) + \frac{1 - c}{1 + c}\sqrt{c' + \left[\frac{1}{2}(1 + c)(y - a)\right]^2}$$

With

$$a = 1.7105249$$
$$b = 0.0301171$$
$$c = 0.0677230$$
$$c' = 0.1889504$$

This function gives the following savings rates by income as a percentage of the reference income :

income	10%	100%	150%	200%
savings rate	11,54%	15,45%	20,64%	29,37%

The proportion of rich is set at $a_2 = 3\%$ and the proportion of poor at $a_1 = 97\%$.

The following results are obtained for μ_1 and μ_2, with an error smaller than 10^{-4}:

$$\mu_1 = 44.18.10^{-2}$$
$$\mu_2 = 0.37.10^{-2}$$

The value $\mu_2 = 0.37.10^{-2}$ represents a rent of 0.37% of the reference income.

The value $\mu_1 = 44.18.10^{-2}$ represents a rent of 44.18% of the reference income.

For μ_0 , we have to compute $\lim_{c \to +}[T(\delta c) - \delta c]$. It turns out that this limit is equal to

$$\lim_{y \to +\infty} [y - S(y)] = a - b$$

Thus $\mu_0 = 1.63.10^{-2}$.

Finally, we verify that the assumptions of section 2 are met, in particular conditions 1 and 2.

6.2. Description of a gradual increase in rent

We examine what happens when μ varies from 0 to a limit value where the equilibrium income of the poor is less than the egalitarian income. This value of μ will be denoted μ_6.

We observe the succession of the following patterns: I, III, IV, V.

We thus begin with a situation close to the case without rent. We obtain the 3 equilibria: lower stable equilibrium k_0', unstable equilibrium k_1, upper stable equilibrium k_2. As mentioned in section 4, as long as μ is small the analysis does not differ much from the case $\mu = 0$ studied in Mabrouk (2016). This means that if the initial capital is insufficient and the propensity to save of the poor is low, the economy may find itself locked in the lower stable equilibrium k_0', which, as long as one is in pattern I, is Pareto-dominated by the upper stable equilibrium k_2.

For example, for $\mu = 0.07.10^{-2}$, the lower stable equilibrium is: $(k_0', c_1, c_2) = (6.52, 6.51, 7.01)$. The upper stable equilibrium is: $(k_2, c_1, c_2) = (11.61,,7.15,157.67)$.[6]

From $\mu = \mu_2 = 0.37.10^{-2}$, we proceed to pattern III. The lower stable equilibrium is then: $(k_0', c_1, c_2) = (6.60, 6.45, 11.46)$. The upper stable equilibrium is: $(k_2, c_1, c_2) = (11.99, 7.17, 167.78)$. This new upper stable equilibrium is better, in the Pareto sense, than the one attained with a lower rent. Thus, the increase of the rent levied on the income of the poor makes it possible to increase not only the income of the rich but also that of the poor! The underlying reason is that rent promotes a better accumulation that improves labor productivity, which, in turn, improves wages.

If μ is still increased, it is observed that starting from $\mu_3 = 0.4.10^{-2}$, we proceed to pattern IV where there is no longer lower stable equilibrium. The risk of falling into poverty[7] no longer exists.

It thus appears that an increase in rent not only improves the economy in the Pareto sense, but also helps to compensate for the possible lack of initial capital which may otherwise threaten to lock the economy in poverty.

If we further increase μ, starting from μ_4 we proceed to pattern V (figure 12). That is, in the plane (k, A) the equilibrium is taken on the curve A_i instead of the curve A_s. Therefore, in the plane (k, c), the equilibrium value of c_2 is now taken on the lower branch of (E_2). The calculation gives $\mu_4 = 1.5.10^{-2}$. The observation shows that at $\mu = \mu_4$ the net income of the poor is maximum. This fact is confirmed by proposition 14. So to speak, μ_4 is the "pro-poor" capitalist rent. This remark is not valid for μ_3 because in this case the upper equilibrium is not realized at k_{v2}.

For the rich, on the other hand, their net income always increases with μ within the limits of the interval of the study (figure 21).

For $\mu = \mu_4$, the unique equilibrium is: $(k_2, c_1, c_2) = (13.13, 7.19, 204.99)$.

From μ_4 on, the analysis of the equilibrium does not change. The average capital at equilibrium continues to increase until exceeding the golden-rule capital k^*. Denote by μ_5 the value of μ beyond which the average capital exceeds k^*. So to speak, μ_5 is the "efficient rent". The calculation gives $\mu_5 \approx 1.56 \cdot 10^{-2}$.

The observed ranking $\mu_3 \leq \mu_4 < \mu_5 < \mu_0$ is in accordance with remarks 6 and 13. The crossing of $\mu_0 = 1.63 \cdot 10^{-2}$ does not change the equilibrium analysis and does not have any particular economic significance. From μ_6 on, the net equilibrium income of the poor falls below egalitarian income. The calculation gives $\mu_6 \approx 16.07 \cdot 10^{-2}$. This level is significantly higher than the pro-poor rent μ_4 and the efficient rent μ_5.

The following figures represent the equilibrium positions for each of the following cases: $0 \leq \mu < \mu_2$, $\mu_2 \leq \mu < \mu_3$ $\mu_3 \leq \mu < \mu_4$ $\mu_4 \leq \mu < \mu_0$ and $\mu_0 \leq \mu$

Arrows indicate the movement of the equilibrium when μ increases:
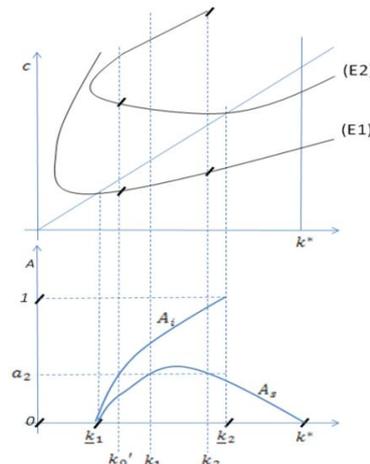
Figure 15. $0 \leq \mu < \mu_2$ (pattern I)
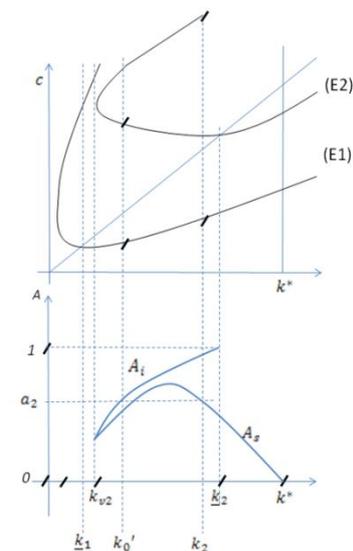
Figure 16. $\mu_2 \leq \mu < \mu_3$ (pattern III)
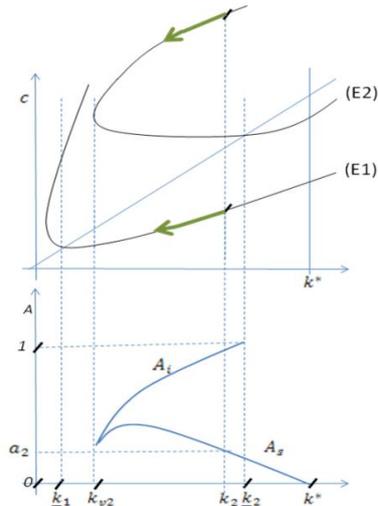
Figure 17. $\mu_3 \leq \mu < \mu_4$ (pattern IV)
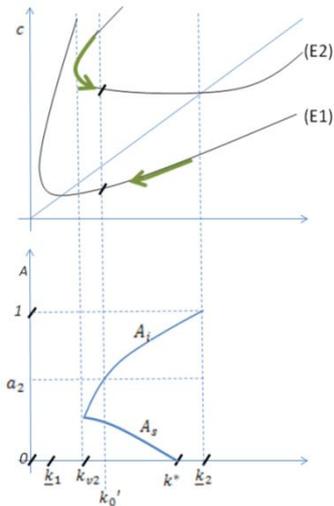
Figure 18. $\mu_4 \leq \mu < \mu_0$ (pattern V)
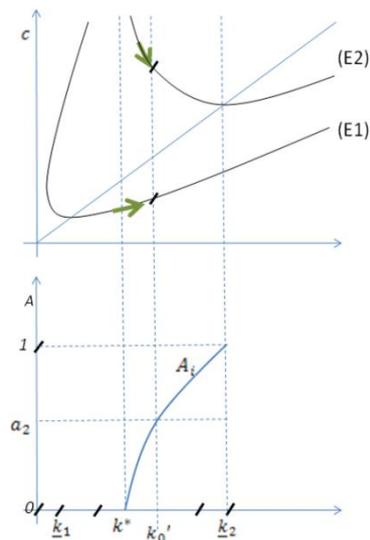
Figure 19. $\mu_0 \leq \mu$ (pattern VI)

6.3. Rent and efficiency

To what extent does rent undermine economic efficiency? Efficiency is conceived here as proximity to the golden rule. The issue is to examine the relationship between rent and the distance between the average capital at equilibrium and the golden-rule capital k^*. We obtain the following trends:
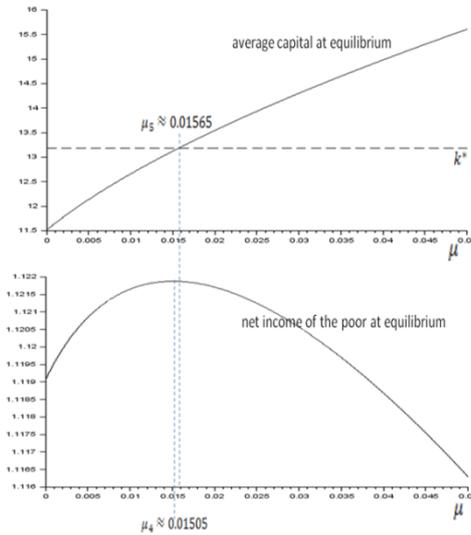
Figure 20.
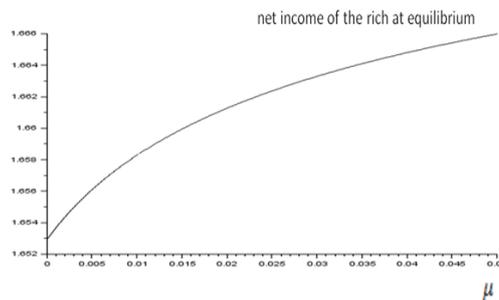
Figure 21.

Thus, the average capital at equilibrium increases with μ. We have no general mathematical proof of this observation. Efficiency is maximal when the average capital at equilibrium reaches k^* for $\mu = \mu_5$. Beyond μ_5, there is overaccumulation of capital. The proximity between μ_4 and μ_5 suggests that it is the poor who bear the cost of overaccumulation because their income begins to decline while the income of the rich continues to grow. We also have no general mathematical proof for the proximity between μ_4 and μ_5.

The plotting of the function $\underline{A}(\mu)$ makes it possible to display the values of μ_2, μ_3, and μ_4, as well as the areas "release from poverty", "Pareto-improvement" and "declining income of the poor":
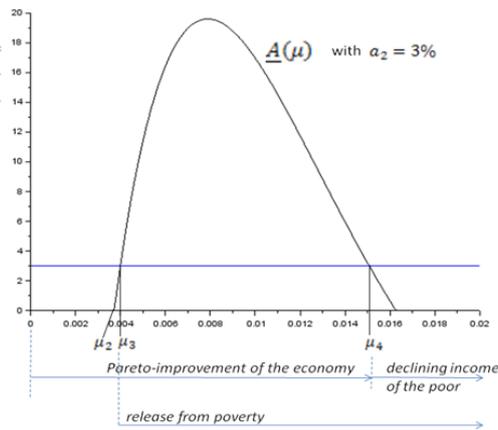
Figure 22.

6.4. Partial release from poverty

As has been shown in Mabrouk (2016), in the case of a zero rent, if one starts with too high a proportion of rich, the only equilibrium is the lower stable equilibrium. Even if the initial capital endowment is high, the economy is caught in a vicious circle of deaccumulation where savings can no longer cover the maintenance costs of a capital stock that has become too high. This was referred to as "Keynesian decline" in Mabrouk (2016), because of a passage from (Keynes 1936) describing a decline caused by the conjunction of an excess of wealth and inequality. In such a case, it is interesting to see what happens when adding a capitalist rent (i.e. rent to the benefit of the rich).

Take $a_2 = 5.5\%$. In this case, with a zero rent, the value of $\overline{A}$ is calculated to be 5.04% (by using the 6 equations given in Mabrouk (2016). The economy declines towards poverty since $a_2 > \overline{A}$. If μ increases, the value of $\overline{A}$ increases. For $\mu = 0.1 \cdot 10^{-2}$, we find $\overline{A} = 5.30\%$. For $\mu = 0.2 \cdot 10^{-2}$ we find $\overline{A} = 5.58\%$. This value is greater than a_2. So there is now an upper stable equilibrium for $\mu = 0.2 \cdot 10^{-2}$. All in all, with a zero rent, we start with the pattern II explained in the following figure; then we go to pattern I as μ increases.
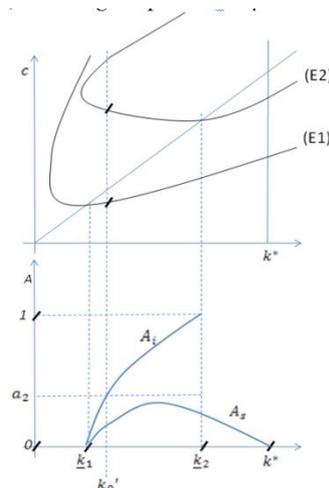
Figure 23: $0 \leq \mu < \mu_3'$ (pattern II)

The value of μ which characterizes the transition from pattern II to pattern I realizes the tangency between the curve $A_s(k)$ and the straight line $(A = a_2)$.

Let's denote it μ_3'. For $a_2 = 5.5\%$, the calculation gives $\mu_3' = 0.17 \cdot 10^{-2}$ and $\mu_3 = 0.76 \cdot 10^{-2}$.

To sum up, for $0 \le \mu < \mu_3'$ we have pattern II. Then, as μ increases, we return to the same evolution as for $a_2 = 3\%$: pattern I for $\mu_3' \le \mu < \mu_2$; pattern III for $\mu_2 \le \mu < \mu_3$; pattern IV for $\mu_3 \le \mu < \mu_4$; pattern V for $\mu_4 \le \mu < \mu_0$; pattern VI for $\mu_0 \le \mu$.

The transition from pattern II to pattern I can be interpreted as a partial release from poverty. Indeed, starting from μ_3', the economy can be released from poverty provided that the initial capital endowment is sufficient. Whereas if the rent crosses the threshold μ_3, the economy is totally released from poverty regardless of the initial capital allocation.
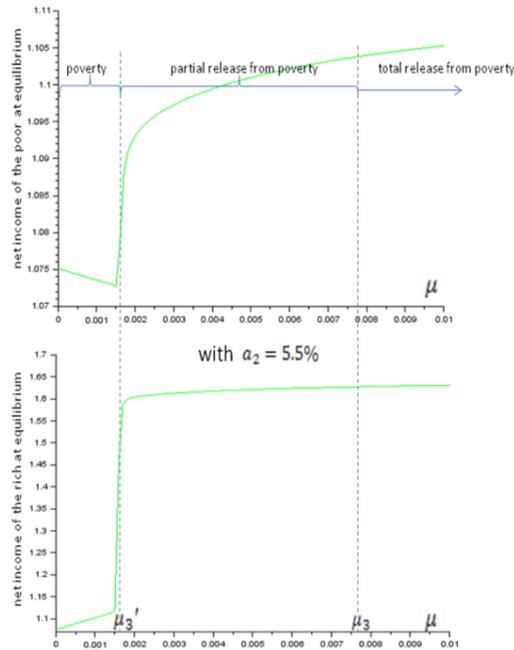
Figure 24.

In conclusion to this section and contrary to immediate intuition, the levying of a rent by the rich class can play a favorable role for the whole economy, including for the poor class.

Moreover, the example studied in this subsection shows that the risk of Keynesian decline can be avoided by means of a rent. Indeed, the rent makes it possible to meet the needs for the maintenance of capital when savings without rent cannot any longer cover them.

However, and more in line with immediate intuition, beyond a certain level of rent ($\mu \approx 1.50\%$ of reference income when $a_2 = 3\%$), the equilibrium income of the poor decreases with the increase of capitalist rent.

7. Variation of a_2

In the case without rent, when a_2 tends to 0, we have seen in Mabrouk (2016) that when the savings of the poor are insufficient, the economy tends towards maximum efficiency whatever the saving function, provided that it is convex. It turns out that this result does not hold in the presence of rent. For example, in the presence of a rent of 0.005, our calculation shows that the average capital at equilibrium clearly exceeds k^* when a_2 tends to 0:
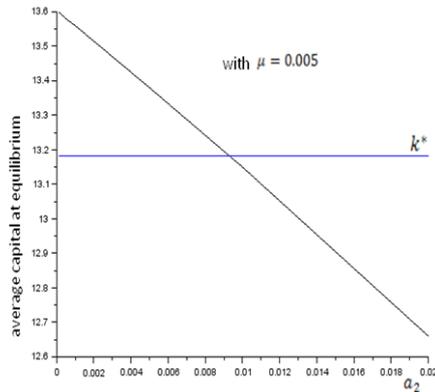
Figure 25.

We now give the evolution of the thresholds $\mu_3', \mu_3, \mu_4, \mu_5$ for a_2 varying from *2%* to *8%* :
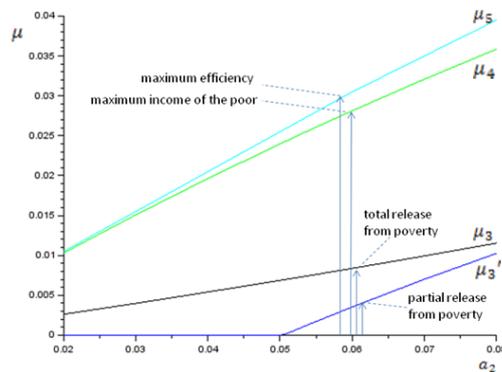
Figure 26.

We read in figure 26 that for $a_2 > 5.04\%$, the economy is doomed to poverty as long as $\mu < \mu_3'$ even if the initial capital endowment is high (Keynesian decline). If μ is in the interval $[\mu_3', \mu_3[$, the economy can be released from poverty provided that it has enough initial capital. If $\mu \geq \mu_3'$, the economy is released from poverty whatever the initial capital.

For $a_2 \leq 5.04\%$, there is no longer any possibility of Keynesian decline. The economy is condemned to poverty only if the initial capital is insufficient. As soon as $\mu \geq \mu_3$, the economy is released independently of the initial capital.

In the following 3 charts, we represent the average capital at equilibrium, the net income of the poor at equilibrium and the net income of the rich at equilibrium as a function of a_2, for different values of μ. These charts show that for $\mu = 0.1 \cdot 10^{-2}$ the Keynesian decline occurs for a_2 between 5.5%, and 6%. For $\mu = 0.5 \cdot 10^{-2}$ the Keynesian decline occurs for a_2 between 6.5% and 7%. The more one increases μ, the more one increases the proportion of rich that the economy is able to bear without falling into decline. This suggests that rent makes it possible to stabilize the accumulation of capital by protecting it from the risk of decline that arises when the proportion of rich becomes high.
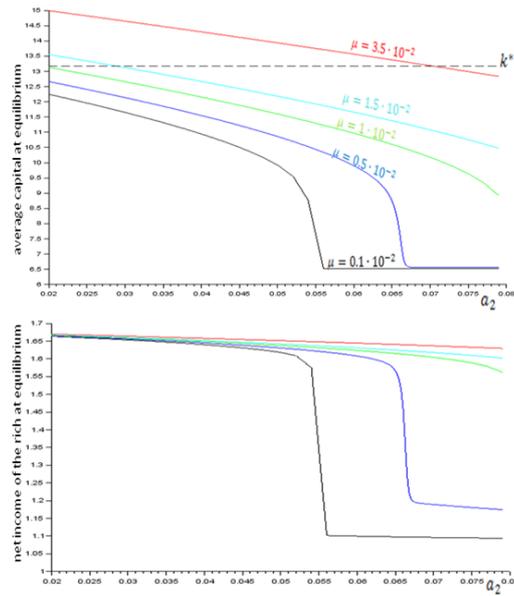
Figure 27.
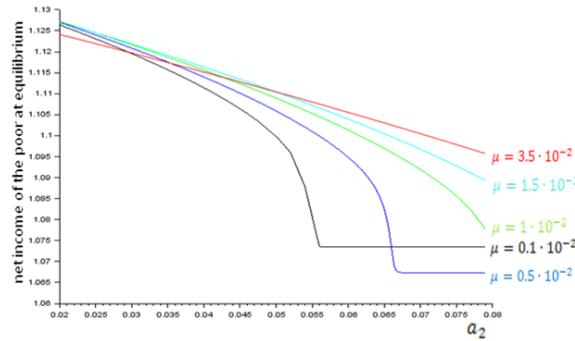
Figure 28.

All other things being equal, the income of the rich always benefits from the increase of the rent which shelters it from Keynesian decline, whereas the outcome for the poor is more nuanced. A high value of rent reduces the income of the poor if the proportion of rich is not excessive. The reason is the cost of overaccumulation that is borne by the poor as seen in subsection 6.3. For the poor, if the proportion of rich is low, it is better to have a low capitalist rent. But if the proportion of the rich is high, it is better to accept a higher capitalist rent in order to rule out the risk of Keynesian decline.

What happens now if, for each value of a_2, the capitalist rent is fixed at its pro-poor level μ_4? The following 2 charts show that everyone wins:
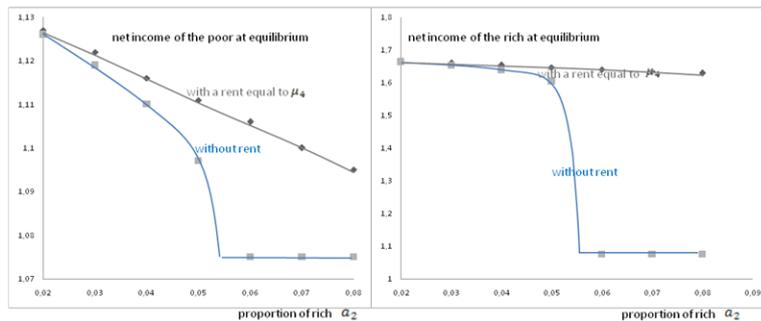
Figure 29.

Note that the value of μ in figure 29 changes for each value of a_2

8. Variation of the social propensity to save

As in Mabrouk (2016), the saving function is modified by introducing a coefficient β in the following way:

$$S_\beta(y) = \frac{1}{\beta} S(\beta y)$$

The variation of the coefficient β represents the variation of the general willingness to save of society. If β increases, this willingness increases and vice versa. For this reason, we call β the "social propensity to save".[8]

If we represent the curve $\underline{A}(\mu)$ of figure 22 for several values of β and with $a_2 = 3\%$, the following figure is obtained:
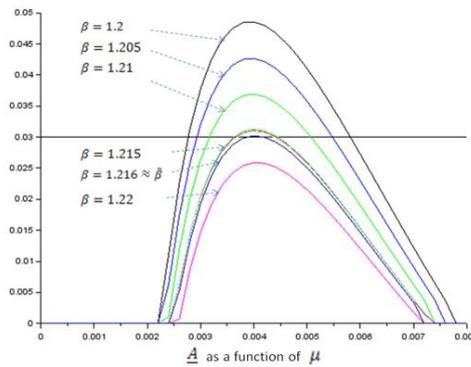
Figure 30.

The two intersections of $\underline{A}(\mu)$ with the horizontal ($\underline{A} = a_2 = 3\%$) are μ_3 and μ_4. If β approaches $\tilde{\beta}$ by lower values, μ_3 and μ_4 approach one another. If β exceeds $\tilde{\beta}$, there is no intersection. This means that if β exceeds $\tilde{\beta}$, there is no longer any risk of Keynesian decline.

We now give the evolution of the thresholds μ_3', μ_3, μ_4 and μ_5 for β varying from 0.8 to 1.25 (with $a_2 = 3\%$).
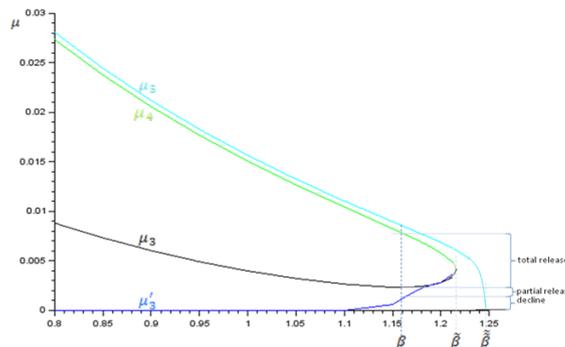
Figure 31.

For $\beta \geq \tilde{\beta}$, the optimal capitalist rent for the poor is 0. This means that when the social propensity to save is high, a rent, even small, is harmful to the poor.

However, we can have $\beta \geq \tilde{\beta}$ and $\mu_5 > 0$. Thus, while harmful to the poor for $\beta \geq \tilde{\beta}$, rent can help improve economic efficiency if it remains below μ_5.

The curve $\mu_5(\beta)$ intersects the x-axis at a point $\tilde{\tilde{\beta}}$. Beyond $\tilde{\tilde{\beta}}$, the economy is overaccumulated whatever the value of the rent. By taking a zero rent, we see that $\tilde{\tilde{\beta}}$ is the solution of the equation $S_{\tilde{\tilde{\beta}}}(f(k^*)) = \delta k^*$. In other words, the egalitarian equilibrium capital $k_0\left(\tilde{\tilde{\beta}}\right)$ is equal to the golden-rule capital k^*. It can be deduced that when the social propensity to save is very high, the rent no longer offers any social advantage. A strong social propensity to save is able to put the economy in the trajectory of a stable accumulation without the help of rent. The only effect of rent would then be to enriching the rich at the expense of the poor. It is only in this case that the effect of rent corresponds to immediate intuition: an unjust and unproductive extortion.

We are now interested with the variation of μ_0 according to β. The value of μ_0 as a function of β is given by the following formula:

$$\mu_0 = a_2 \left(\lim_{c \to \infty} \left[T_\beta(\delta c) - \delta c \right] - \left[f(k^*) - \delta k^* \right] \right)$$
$$= a_2 \left(\frac{1}{\beta} \lim_{c \to \infty} \left[T(\delta c) - \delta c \right] - \left[f(k^*) - \delta k^* \right] \right)$$

For $\beta = 0.8$, we obtain $\mu_0 = 2.89 \cdot 10^{-2}$. For $\beta = 1.2$, we obtain $\mu_0 = 0.79 \cdot 10^{-2}$. It is observed that for any value of β, μ_0 is greater than μ_5. This is consistent with remark 6.

The following 3 charts show the average capital at equilibrium, the net income of the rich at equilibrium, and the net income of the poor at equilibrium as functions of the social propensity to save. These charts confirm that the increase of rent prevents Keynesian decline and that it is always profitable to the rich, all other things being equal. For the poor, we see that if the social propensity to save is strong, a capitalist rent, however small, is unfavorable to them. But if the social propensity to save is low, it is profitable for them to accept a certain level of capitalist rent. This allows for accumulation and maintenance of capital which would otherwise be impossible because of the weakness of the social propensity to save.
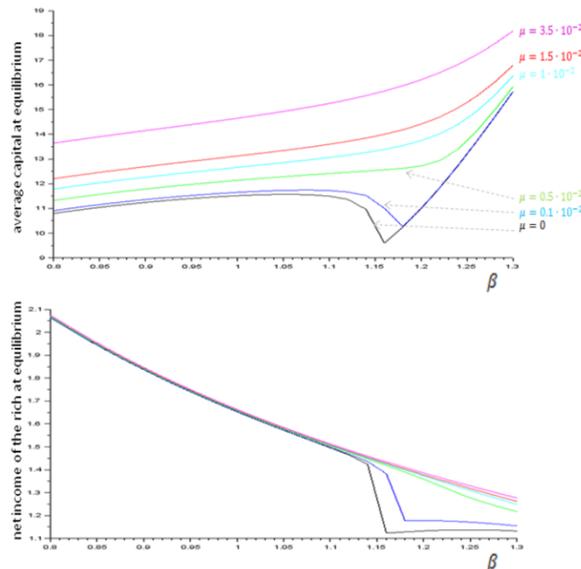
Figure 32.
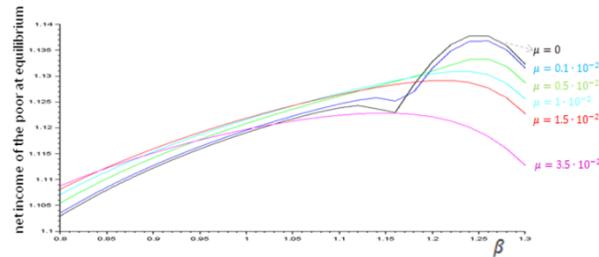
Figure 33.

9. Conclusion

The following lessons can be drawn from this study:

1- When capitalist rent is low, it can improve the poor's income. Indeed, not only does it allow a Pareto-improvement of the economy, but also, it may unlock the economy from under-accumulation trap. If the proportion of rich is small, this unlocking can even occur while capital endowment is very insufficient.

2- The level of capitalist rent that makes the situation of the poor worse than it would be under egalitarianism is significantly higher than that maximizing overall efficiency (efficient rent) or maximizing the income of the poor (pro-poor rent).

3- Capitalist rent makes it possible to stabilize capitalism by avoiding the risk of deaccumulation caused by an insufficiency of savings to cover the maintenance of a too large capital (Keynesian decline). This risk, highlighted in Mabrouk (2016), appears particularly in the context of an increase in the proportion of rich. In such a case, rent-seeking behavior might be individually and collectively beneficial.

4- Capitalist rent begins to be clearly harmful to the poor only if the economy is close to the stage of overaccumulation. In other words, as long as net productivity of capital is positive, moderate capitalist rent does not impoverish the poor. It enriches them by encouraging the accumulation of capital which increases wages. However, it should be kept in mind that this hold under our neoclassical assumption that wages remain linked to the productivity of labor. When the rent reaches a level such that the economy becomes overaccumulated, it is the poor who bear the cost of overaccumulation.

5- A strong social propensity to save can put the economy on a good trajectory of accumulation without recourse to capitalist rent. For the poor, moderate capitalist rent makes it possible to palliate the weakness of the social propensity to save. But it becomes detrimental to them if the social propensity to save is strong.

These lessons rely of course on the simplifying assumptions of our model: no money, only one good, no technical progress, no uncertainty, and most importantly the assumption of a rigid saving behavior not related to the position in the accumulation trajectory. The main difference between this assumption and the standard intertemporal optimization model is the persistence of a strong propensity to save for high incomes in periods when greater consumption would have been socially preferable. Nevertheless, we believe that this type of behavior, although rigid, is more realistic than intertemporal optimization because the latter does not capture the game between capitalists who, at a certain stage of accumulation, are under the threat of deaccumulation because of the decline in the productivity of capital. It is likely that this threat contributes to a high propensity to save at the wrong time. There is much to gain from studying this issue in the context of a dynamic game.

Notes

[1] See for example: (Oxfam report "Even It Up" 2014).

[2] For more precision on the meaning of the word "rent" in this context, see Stiglitz (2015b) page 7.

[3] Murphy, Shleifer, & Vishny (1993) analyzed the effect of "rent-seeking" behavior in terms of efficiency and economic growth. However, their approach differs from ours because, on the one hand, it considers rent-seeking as a productive activity in its own right, and on the other it does not place the question of rent in a dynamic perspective of capital accumulation.

[4] For the proposed numerical application, we will see that this condition is not limiting since the value of μ_1 is more than 44%. It goes far beyond the other critical values of μ that our analysis reveals.

If $\mu > \mu_1$ the curve (E_1) would divide into two branches, one above the line $(k = c)$ and the other beneath. The interesting branch is that which is below, as in the case $\mu \leq \mu_1$. We will not deal here with the case $\mu \geq \mu_1$

[5] The saving function is slightly modified so as to ensure perfect equality $S(0) = 0$. This is because exact equality is required for the calculation of the positions of the curves for high values of μ.

[6] The values of capital are given with an error smaller than 10^{-2} and the values of rents are given with an error smaller to 10^{-4}.

[7] I use the terminology "poverty" to describe a state of general under-accumulation.

[8] As in Mabrouk (2016), we draw the reader's attention to the fact that the variation of the coefficient β alone can not represent all the possibilities of modifying the profile of the willingness to save. For example, one can conceive of an increase in the willingness to save among the poor and simultaneously a decrease in this willingness among the rich. Such a modification is not captured by the parameter β and is not considered in the present study.

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