Mathematical Analysis of Income Per Capita in the United Kingdom

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Abstract. Industrial Revolution did not boost economic growth and the growth of population even in the United Kingdom, the centre of this revolution. On the contrary, from around 1850, economic growth and the growth of population started to be diverted to slower trajectories. The new trend describing the growth of income per capita (Gross Domestic Product per capita, i.e. GDP/cap) is now analysed using Maddison’s data. Two numerical solutions and two analytical solutions of the differential equation describing the growth rate of income per capita are presented. It is demonstrated yet again that even strong fluctuations in the growth rate do not change the shape of growth trajectories. Thus, contrary to the common misconception, even strong fluctuations in the growth rate cannot be used as the evidence of the existence of Malthusian stagnation because they do not change the mechanism of growth. Such strong fluctuations can at best produce only small and negligible ripples in growth trajectories. Our analysis shows also that the current growth of income per capita in the UK follows an unsustainable trajectory.

Keywords. United Kingdom, Income per capita, Growth rate, Future growth

JEL. A10, B41, C02, C12, C20, C50, O10.

1. Introduction

According to the generally accepted interpretations, the alleged long epoch of the so-called Malthusian stagnation in the economic growth and in the growth of human population was followed by a rapid increase, which is claimed to have been caused by modern progress reflected in and coinciding with the Industrial Revolution. The United Kingdom was at the centre of the Industrial Revolution and consequently, this postulated transition from stagnation to growth should be most clearly demonstrated in this country.

We have already shown that the currently accepted interpretations are incorrect (Nielsen, 2014, 2015a, 2016a, 2016b, 2016c, 2016d, 2016e, 2016f, 2016g, 2016h, 2016i, 2016j, 2016k). Within the range of analysable data, epoch of Malthusian stagnation did not exist in the economic growth and in the growth of population, global or regional. Likewise, the Industrial Revolution, 1760-1840 (Floud & McCloskey, 1994) did not boost the growth trajectories.

In particular, we have demonstrated (Nielsen, 2016k) that even in the United Kingdom, where the effects of the Industrial Revolution should be most clearly demonstrated, there was no boosting in the economic growth and in the growth of population. On the contrary, shortly after the Industrial Revolution, economic

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growth and the growth of population started to be diverted to slower trajectories. We have also demonstrated that within the range of the mathematically analysable data, the mythical epoch of Malthusian stagnation did not exist in the United Kingdom.


Our aim now is to investigate the new trend of income per capita in the UK, the trend which commenced relatively recently when the economic growth started to be diverted from its historical, linearly modulated hyperbolic trajectory (For its definition see Nielsen, 2015a). We shall first show how to describe growth by using the differential equation defining the growth rate. This part of the analysis will show that contrary to the generally expected outcomes, fluctuations and long-term variations in the growth rate have negligible effect on the growth trajectories. We shall then study the future growth of income per capita.

2. Overview

Distributions describing the growth of population and economic growth in the UK based on using Maddison’s data (Maddison, 2010) are shown in Figure 1-3.

**Figure 1.** Growth of population in the UK between AD 1 and 2008. Growth was hyperbolic between AD 1 and 1850. From around 1850, towards the end of the Industrial Revolution, the growth of population started to be diverted to a slower trajectory. Industrial Revolution had no impact on shaping the growth trajectory.
Figure 2. Economic growth (as described by the GDP) in the UK. The growth was hyperbolic between AD 1 and 1600 and again (but a little slower) between AD 1600 and 1850. From around 1850, the growth started to be diverted to a slower trajectory. Within the range of analysable data, i.e., from AD 1, the mythical epoch of stagnation did not exist. Economic growth was steadily increasing. Industrial Revolution did not boost the economic growth. There was no escape from the Malthusian trap because there was no trap.

Figure 3. Growth of income per capita (GDP/cap) in the UK between AD 1 and 2008. The GDP data follow closely the empirically-determined linearly-modulated hyperbolic distributions (defined in Nielsen, 2015a). Industrial Revolution did not change the growth trajectory. From around 1850, the growth of the GDP/cap started to be diverted to a slower trajectory.

Population and the Gross Domestic Product (GDP) were increasing hyperbolically. Contrary to the currently accepted interpretations, there was no Malthusian stagnation and the Industrial Revolution had no impact on the economic growth and on the growth of population even in the United Kingdom, the very centre of this revolution.

We should notice that at the time of the Industrial Revolution, economic growth and the growth of population in the United Kingdom were close to escaping to infinity. It was most fortunate that natural processes did not comply with the imagined interpretations of the growth mechanism. Any boosting by the Industrial Revolution would have been catastrophic.

While the growth of population was following a single hyperbolic trajectory, the growth of the GDP experienced a transition around AD 1600 from a fast to a slower hyperbolic trajectory. This transition is reflected in the income per capita (GDP/cap) shown in Figure 3. Industrial Revolution did not boost the growth of

population or the economic growth. From around 1850, economic growth and the
growth of population started to be diverted to a slower, non-hyperbolic, trajectory.
This simultaneous transition in the growth of population and in the growth of the
GDP is reflected in a clear transition in the income per capita (GDP/cap). It is the
purpose of this publication to investigate this new trajectory.

3. Mathematical method
Our analysis is based on the examination of the growth rate (Nielsen, 2015b):

\[
\frac{1}{S(t)} \frac{dS(t)}{dt} = R_e(t)
\]

where \( S(t) \) is the size of the growing entity and \( R_e(t) \) is the empirically-determined
growth rate. In the case discussed here, the size of the growing entity is the GDP/cap.

There are two ways of solving this equation: numerical or analytical. If the empirically-
determined growth rate \( R_e(t) \) can be described analytically, then

\[
S(t) = \exp\left[ \int f(t) dt \right]
\]

where \( f(t) \) is the analytical representation of \( R_e(t) \).

If \( R_e(t) \) cannot be represented by a simple mathematical function, as in the case of
randomly-fluctuating growth rate, then the eqn (1) has to be solved numerically. We also
have to solve the eqn (2) numerically if the integration of the function \( f(t) \) leads to
computational problems, such as when \( S(t) \) has to be expressed by an infinite series. We
shall now use both of these methods, analytical and numerical to describe the growth of
income per capita in the UK and to predict growth.

4. Mathematical analysis
Four representations of the growth rate of income per capita (GDP/cap) in the
UK between AD 1830 and 2008 are shown in Figure 4. They are (1) \( R(Direct) \)
calculated directly from the GDP/cap data; (2) \( R(Refined) \) calculated using the
GDP/cap data and interpolated gradients; (3) calculated using the best polynomial
fit to \( R(Refined) \) represented in this case by a sixth-order polynomial; and (4)
calculated by using linear fit to \( R(Direct) \). Virtually the same linear distribution
was obtained by fitting \( R(Refined) \).
Figure 4. Four representations of the growth rate of income per capita (GDP/cap) in the United Kingdom between AD 1830 and 2008. R(Direct) is the growth rate calculated directly from the GDP/cap data. R(Refined) is the growth rate calculated using the GDP/cap data and interpolated gradients. $f(t)$ - Polynomial is the sixth-order polynomial fitted to R(Refined) and $f(t)$ - Linear is the linear fit to R(Direct).

We shall now use all these four representations of the growth rate to describe the GDP/cap distribution. We shall present two numerical solutions of the eqn (1) by using $R_{c}(t) = R(Direct)$ and $R_{c}(t) = R(Refined)$. We shall also present two analytical solutions $f(t)$ represented by a sixth-order polynomial or by the linear function. Finally, we shall use the linear representation of the growth rate to predict growth of income per capita.

4.1. Describing the growth trajectory

Two numerical solutions of the eqn (1) are presented in Figure 5. They are so close to the data that in order to see the difference between them we have to look at a magnified section (Figure 6) in the region of large fluctuations of $R(Direct)$ (see Figure 4).

Results presented in Figures 5 and 6 show that the two numerical integrations of the eqn (1) give excellent description of data. However, while the numerical integration using $R_{c}(t) = R(Refined)$ reproduces the general trend of the GDP/cap distribution, the calculation based on using $R_{c}(t) = R(Direct)$ reproduces the fine structure.

We can now understand the origin of the fine structure, which can be seen in Figure 5, and even more clearly in Figure 6. These small ripples are caused by strong fluctuations in the growth rate. However, it is important to notice that even strong fluctuation in the growth rate do not change the growth trajectory.

It is incorrect to claim that fluctuations in the growth rate represent evidence of the existence of Malthusian stagnation. They do not. Whatever might be their origin, they have no tangible effect on the growth trajectories and consequently on the mechanism of growth. Fluctuations in the growth rate can be neglected when trying to understand the mechanism of growth.
Figure 5. The growth of income per capita (GDP/cap) in the UK between 1830 and 2008. Data of Maddison (2010), are reproduced by carrying out numerical integration of the eqn (1) using $R_t(t) = R(Direct)$ or $R_t(t) = R(Refined)$, both growth rates displayed in Figure 4. Both numerical calculations give good representation of data.

Figure 6. Magnified section of the GDP/cap distributions showing the difference between the results of two numerical integrations of the eqn (1). These calculations explain the origin of the fine structure of GDP/cap distribution.

We shall now turn our attention to the analytical solutions of the eqn (1) given by the eqn (2). In Figure 7 we show two such solutions, using $f(t)$ representing either the best, sixth-order polynomial, fit to $R(Refined)$ or the best linear fit to $R(Direct)$. In order to examine the differences between these two solutions, we are displaying data every 10 years.

Figure 7. Two analytical solutions of the differential eqn (1) compared with the GDP/cap data (Maddison, 2010). While the solution obtained using $f(t)$ represented by linear function reproduces the general trend of the GDP/cap distribution, the solution corresponding to $f(t)$ represented by the sixth-order polynomial reproduces the gentle oscillations around the general trend.

Results presented in Figure 7 show that the solution based on using the \( f(t) \) represented by a linear function reproduces the general trend of the GDP/cap distribution but the calculations based on using the sixth-order polynomial, which reproduces the oscillating behaviour of \( R(Refined) \), reproduces also the gentle oscillations of the GDP/cap around the general trend.

We can now understand the origin of these gentle oscillations in the GDP/cap distribution: they are generated by the long-term oscillations of the growth rate. These oscillations are present in \( R(Direct) \) but they are obscured by strong fluctuations. However, they are revealed in \( R(Refined) \), which is calculated using interpolated gradient.

Thus, in summary, combining results presented in Figures 5-7, we can see that small ripples in the time-dependent distributions describing growth, if present, reflect strong fluctuations in the growth rate, while gentle oscillations around the prevailing trend reflect the long-term oscillations in the growth rate.

We have now shown how the interpretations of the mechanism of growth can be simplified. We do not have to worry about the strong fluctuations or about the long-term oscillations in the growth rate. We can concentrate our attention on the general trend of the time-dependent distributions and on simple mathematical representations of the growth rate.

Of course, if we want to go a step further and to try to explain the origin of minor forces, which have no impact on the general trend, we would have to study the oscillations or minor ripples in the time-dependent distributions. Maybe such studies could lead to some interesting discoveries but they would have no impact on explaining the prevailing mechanism of growth.

4.2. Predicting growth

We can now use the GDP/cap data between AD 1830 and 2008 to predict economic growth. We can see that the growth is not exponential because the best linear fit to the growth rate is not constant. The linear fit,

\[
f(t) = a_0 + a_1 t,
\]

shown in Figure 4 is described by parameters \( a_0 = -8.964 \times 10^{-2} \) and \( a_1 = 5.549 \times 10^{-4} \). The gradient is small but positive, which means that the growth rate is steadily increasing. The growth rate around AD 1830 was about 1% but by 2000 it increased to around 2%. By 2050, it is projected to increase to 2.2% and by 2100, to 2.5%.

The predicted growth is faster than the corresponding exponential growth fitting the same data. Any exponential growth becomes unsustainable after a certain time but the growth of income per capita in the UK is going to become unsustainable even faster than the corresponding exponential growth. The predicted growth is shown in Figure 8 and in Table 1.
Figure 8. The projected growth of income per capita in the United Kingdom.

Sustainability of economic growth is defined not only by the availability of natural resources but also by the associated stress to maintain a given growth. We have defined the relative stress factor for the growth of the GDP (Nielsen, 2015c). We can use the same definition for the growth of income per capita. Thus, the relative stress factor $\sigma$ for the growth of income per capita can be defined as

$$\sigma \equiv \frac{\left( \frac{GDP}{cap} \right)_t}{\left( \frac{GDP}{cap} \right)_{t_0}}$$

(4)

where $\left( \frac{GDP}{cap} \right)_t$ is the income per capita at a certain time $t$ and $\left( \frac{GDP}{cap} \right)_{t_0}$ is the income per capita at a certain, fixed time $t_0$.

Table 1. Projected growth of income per capita in the United Kingdom and the associated relative stress factor.

<table>
<thead>
<tr>
<th>Year</th>
<th>GDP/cap</th>
<th>R</th>
<th>$\sigma$</th>
<th>Year</th>
<th>GDP/cap</th>
<th>R</th>
<th>$\sigma$</th>
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</thead>
<tbody>
<tr>
<td>2000</td>
<td>22,031</td>
<td>1.95</td>
<td>1.00</td>
<td>2080</td>
<td>125,268</td>
<td>2.39</td>
<td>5.69</td>
</tr>
<tr>
<td>2015</td>
<td>29,717</td>
<td>2.04</td>
<td>1.35</td>
<td>2090</td>
<td>159,536</td>
<td>2.45</td>
<td>7.24</td>
</tr>
<tr>
<td>2020</td>
<td>32,924</td>
<td>2.06</td>
<td>1.49</td>
<td>2100</td>
<td>204,291</td>
<td>2.50</td>
<td>9.27</td>
</tr>
<tr>
<td>2030</td>
<td>40,579</td>
<td>2.12</td>
<td>1.84</td>
<td>2110</td>
<td>263,033</td>
<td>2.55</td>
<td>11.94</td>
</tr>
<tr>
<td>2040</td>
<td>50,289</td>
<td>2.17</td>
<td>2.28</td>
<td>2120</td>
<td>340,520</td>
<td>2.61</td>
<td>15.46</td>
</tr>
<tr>
<td>2050</td>
<td>62,662</td>
<td>2.23</td>
<td>2.84</td>
<td>2130</td>
<td>443,246</td>
<td>2.66</td>
<td>20.12</td>
</tr>
<tr>
<td>2060</td>
<td>78,508</td>
<td>2.28</td>
<td>3.56</td>
<td>2140</td>
<td>580,121</td>
<td>2.72</td>
<td>26.33</td>
</tr>
<tr>
<td>2070</td>
<td>98,899</td>
<td>2.34</td>
<td>4.49</td>
<td>2150</td>
<td>763,419</td>
<td>2.77</td>
<td>34.65</td>
</tr>
</tbody>
</table>

GDP/cap in the 1990 International Dollars; $R$ – the growth rate of the GDP/cap, in per cent; $\sigma$ - the relative stress factor, in per cent.

The relative stress factor in 2015 was only 35% higher than in 2000. A 35% greater effort was required to keep the economy growing along this new trajectory. By 2050, the stress factor is projected to increase to 2.84. Economic output per year will have to be almost three times as high as in the year 2000 to keep the economy growing along the same trajectory. Such a large stress might be already hard to tolerate. By the end of the current century, the annual economic output per year will have to be about 9 times as high as in the year 2000 and by 2150 it would have to be about 35 times as high. Even with unlimited natural resources, there will come a time when such a large economic output will be physically impossible to achieve and the economic growth will either have to be diverted to a new trajectory or it will collapse.
The general drive everywhere, not only in the UK but also in other countries, is to keep the economic growth rate increasing or constant. This is a serious mistake. Even with a constant growth rate, which describes exponential growth, such economic growth will become, at a certain stage, impossible to maintain, even if we had unlimited natural resources. To make the economic growth safe and secure, the growth rate should be now slowly decreasing, not only in the UK but also globally (Nielsen, 2015c).

5. Summary and conclusions

We have carried out the analysis of Maddison’s data (Maddison, 2010) describing income per capita (GDP/cap) in the United Kingdom between 1830 and 2008. Our analysis is based on solving differential equation describing the growth rate. We have presented two numerical and two analytical solutions of this equation. We have explained the origins of various features of the time-dependent GDP/cap distribution.

We have demonstrated that strong fluctuations in the growth rate do not change the growth trajectory. They can, at best, be reflected only as just small ripples along the prevailing trend. It is incorrect to interpret even strong fluctuations in the growth rate as the evidence of the existence of Malthusian stagnation because these fluctuations have no impact on shaping growth trajectories.

Long-term oscillations in the growth rate can be reflected as small oscillations of the growth trajectory. They are also unlikely to affect the general trend of growth. The mechanism of growth is determined by the prevailing trend of the growth trajectory.

In order to study the mechanism of growth or to predict its future there is no need to worry about reproducing mathematically the details of the corresponding growth rate. Random fluctuations and long-term oscillations in the growth rate can be neglected and the growth rate can be reproduced by a simplest function. Often, it is possible to do it by using linear functions. Indeed, using the simplest descriptions of the growth rate is most acceptable.

Our analysis demonstrated that the current economic growth in the UK is unsustainable even if supported by unlimited natural resources, because after a certain time it will be impossible to maintain the ever-increasing output. At a certain time in the future, economic growth will have to start to be diverted to a slower trajectory or it will be likely to collapse.

The same problem applies globally (Nielsen, 2016c). Global economic growth should now, or soon, be characterised by a slowly decreasing growth rate. The example of the economic growth in Greece shows that rapid decrease or increase in the growth rate can lead to catastrophic results (Nielsen, 2015d).
References


Turkish Economic Review


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