Journal of
Economics and Political Economy
www.kspjournals.org
Volume 3 September 2016 Issue 3

Theory of the Firm: A Reformulation with Primary Factors of Production and Procurement of Ingredient Inputs

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Abstract. This paper reformulates the neoclassical theory of the firm by distinguishing two types of inputs: (1) the primary factors of production (labor, capital, etc.) and (2) ingredient inputs (intermediate goods, raw materials, and services). The production function is defined on the space of the primary factors while ingredient inputs, as required by production technologies, are procured externally from other firms. Firms maximize profits subject to the production function as well as to the ingredient input requirement functions. We analyze how the optimal level of production and the optimal employment of factor services are determined when the cost of the acquisition of ingredient inputs is counted explicitly as part of the total cost of production. The first order condition of profit maximization requires that the marginal value-added product of an employed primary factor be equal to its price, and the second order condition is stated in terms of the negative definiteness of the Hessian of the value-added function. Cost minimization requires that the marginal cost of production be equal to the sum of an incremental cost of factor services and an incremental cost of ingredient inputs that are procured. The optimum level of production and the optimal use of the primary factors both respond to changes in the prices of ingredient inputs. The paper also shows: the zero degree homogeneity of factor demand and output supply functions, the linear homogeneity of the value-added function, Shephard's lemma, the interpretation of the Lagrangian multiplier in cost minimization, the nonlinearity of the iso-cost surfaces, and the concavity of the cost function.
Keywords. Primary factors, Ingredient inputs, Production function, Value-added function, Marginal value-added product.
JEL. D01, D21, D24.

1. Introduction

Ingredient inputs or intermediate goods refer to all inputs other than the primary factors of production. They include parts, raw material, power, services, and any other intermediate good. The trend toward external procurement of such inputs has accelerated with globalization, and now it is common practice for high tech product firms, such as Boeing, Apple, Microsoft, and numerous others, to outsource the necessary intermediate goods or tasks. In fact, such outsourcing has spread across almost all industries by now, with world having turned into a global network of producers of final goods and suppliers of intermediate goods under the division of production tasks through functional and spatial fragmentation (Sydor,

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2011; Globerman, 2011; De Backer & Yamano 2011; Grossman & Rossi-Hansberg, 2008). With such division of the tasks, there has been a new development in economic research to explain how the total value-added created in any economy is affected by the development of value or supply chains and the complementarity in the division of the tasks through such chains. A fundamental question, asked in various forms, has been if and how the share of trading in intermediate goods and the vertical division of the tasks affect the total factor productivity, aggregate income, and economic growth (Hulten, 1978; Grossman & Rossi-Hansberg, 1979; Ciccone, 2002; Jones, 2011; Kurz & Lengermann, 2008; Peng, Riezman, & Wang, 2013; Moro, 2012). At a more disaggregated level, a similar question has been addressed on the impact of imported intermediate goods and trade liberalization on the product growth and firm productivity (Halpern, Koren, & Szeidl, 2015; Goldberg, Khandelwal, Pavcnik, & Topalova, 2010); Khandelwal & Topalova, 2011).

One popular form of the production functions has been of the CES form, which reduces to the Cobb-Douglas function when the elasticity of substitution is 1, to a linear function when the elasticity is infinite, and to a Leontief (minimum) function when the elasticity goes to zero. The Cobb-Douglas form has been particularly popular in the literature on the effect of intermediate goods on total factor productivity (Moro, 2012; Jones, 2011). Since intermediate goods enter the production process basically as a complement, changing such goods alone, without simultaneously increasing primary factors (such as labor and capital), would not yield additional output. In fact, this is the reason why industrialization involving intensive use of intermediate goods and the division of the tasks will create strategic complementarities that can possibly account for the presence of the multiplier effect and make a difference in the total factor productivity and economic growth (Ciccone, 2002; Jones, 2011). It is also what makes the Leontief input-output analysis operational (Leontief, 1936), which essentially keeps the proportion of inputs at a constant ratio. The complementarity of inputs implies that the differential approach to the theory of production runs into a difficult conceptual problem since the production function that includes all inputs as its arguments is not differentiable with respect to each of the inputs.

This problem has been sidestepped, with no explicit distinction made between primary factors that are hired by the firm and ingredient inputs or intermediate goods that are procured externally, despite the fact that the theory, as developed by Hicks (1946) and Samuelson (1947), has been elaborated along the duality between production technologies on the one hand and cost structures and profit functions on the other (Shephard, 1953; 1971; Uzawa, 1964; McFadden, 1978; Diewert, 1973; 1974). This paper is an attempt to close this gap by reformulating the traditional theory of the firm by directly addressing the complementarity issue, and by showing how to resolve the conceptual difficulty that the theory runs into when both of these input classes are entered into the production function as independent arguments. It would not be an exaggeration to say that, despite several attempts to analyze value-added functions or value-added production functions (McFadden, 1967; Khang, 1971; Arrow, 1974; Bruno, 1978; Diewert, 1978), the question still remains at large on how ingredient inputs should be treated in the theory of production and cost along with the primary factors of production.[1]

The primary factors (labor, capital, land, managerial talent) act on intermediate goods, raw materials, and services that are, in many cases, acquired now from external sources in creating the value-added – Ferguson (1969) called them ‘ingredient inputs’ several decades ago. The two categories, therefore, cannot be independent arguments of the production function. It is obvious that firms cannot produce output by simply procuring more ingredient inputs unless additional services of primary factors are employed that can act on them, nor are they able to produce more by simply increasing the services of primary factors unless additional ingredient inputs are available. Hence, for any given primary factor combination and the production technologies that accompany it, the firm must make decisions on how much to procure of the ingredient inputs. Due to this relationship, it is not permissible to enter both types of inputs into the production function pretending that they are independent arguments and to assume that the usual regularity condition on the marginal product of each input and the Hessian of the production function is met. If only the primary factors are allowed as independent arguments of the production function, this function has to be defined properly with the need of the ingredient inputs fully taken into account, and the value of these inputs should not be left out in the computation of profit and cost. Otherwise the neoclassical theory of production and cost would lose much of its operational advantage (Samuelson, 1947; Ferguson, 1969; and Mas-Colell *et al.*, 1995).[2]

It is true that the primary factors may substitute some of the ingredient inputs if the firm chooses to procure certain ingredient inputs internally through vertical integration. Such decision depends on the relative advantage of internal over external procurement, but it only changes the composition of such procurement. No single firm is self-sufficient as far as ingredient inputs are concerned. For this reason a careful distinction should be made between primary factors and ingredient inputs in specifying the production technologies. From a macroeconomic standpoint, we hear an argument that what an aggregate production function represents is the maximum value-added that can be produced from a primary factor combination. But, if so, the production of the value-added should depend critically on the availability of the ingredient inputs; in the extreme case, in which the supply of the latter is cut off, creation of the value-added will be seriously hampered if no substitute is found. A country buys many ingredient inputs, some in large quantities, from abroad through the division of production tasks. Hence, the amount of the value-added that can be created by the country's primary factors of production depends crucially on those inputs acquired from foreign sources (the energy goods such as oil and gas, rare earth metals, computer chips, and many other intermediate goods). Real shocks in the supply of these inputs in the world market have severe impacts on many nations relying on them. Thus, even at a macro level where only the value-added created by the primary factors is counted as the product, it is presumptuous to assume away the role of ingredient inputs acquired from foreign sources (see Cobbold (2003) for a comparison of gross output and value-added methods of productivity estimation). In either case, micro or macro, ingredient inputs should be kept separate from primary factors in representing the production technologies.[3]

We proceed with the premise that the acquisition of ingredient inputs from external sources depends on the nature and the kind of the production technologies chosen by the firm, which are co-determinable with the choice of primary factor combinations. This implies that the firm, in maximizing its profits, chooses the best primary factor combination from available technologies knowing how this choice determines the choice of required ingredient inputs.

This paper is organized as follows: In section 2, production function and profit function are defined with ingredient inputs distinguished from primary factors. The conceptual problem will be elucidated that arises when all types of inputs, primary or ingredient, are inserted indiscriminately into production functions as independent arguments. In sections 3 and 4, the profit maximization problem is looked at when the cost of acquisition of ingredient inputs is counted as part of the cost of production. Several observations are made on the interpretation of the first-order conditions, the negative definiteness of the Hessian of the value-added function, the homogeneity of factor demand and output supply functions, and the homogeneity of the value-added function. Section 5 analyzes cost minimization. Again, observations will be made on the interpretation of the Lagrangian multiplier, the linearity of iso-cost surfaces, Shephard's lemma, and the concavity of the cost function. Section 6 touches on how the Solow residual in growth accounting may reflect the variation of the cost of ingredient inputs acquired from foreign sources. Section 7 concludes the paper.

2. Production Function

To elucidate the conceptual difficulty, consider the case in which a firm produces a single output x from n inputs, denoted as a vector *v*, according to a production function $x = f(v)$, which is assumed to be twice continuously differentiable. The input space is the nonnegative orthant of the Euclidean n space, R_0^n, and no distinction is made between primary factors and ingredient inputs. Under the assumption that input and output markets are competitive, the firm maximizes its profit:

$$Max\ \Pi(v) = pf(v) - wv,\ v \geq 0 \tag{1}$$

Where $\Pi(v)$ is the profit as a function of v; p is the price of output; w is a price vector of an input vector v. If the solution is assumed to be interior, the first order condition is given by

$$pf_i - w_i = 0,\ i = 1,2,\ldots,n\ . \tag{2}$$

where f_i is the partial derivative of $f(v)$ with respect to the *i*-th input. The second order condition is stated in terms of the negative definiteness of the Hessian matrix $H_\Pi(v)$ of $\Pi(v)$. Here this condition is equivalent to the negative definiteness of the Hessian matrix $H_f(v)$ of the production function $f(v)$.

$$H_\Pi(v) = pH_f(v) = p\left[f_{ij}\right] \text{ is negative definite.} \tag{3}$$

By the second order condition, the Jacobian of $\Pi(v)$ is nonzero; hence, by the implicit function theorem, the optimal levels of inputs can be solved as functions of p and w. This is the neoclassical theory in a nutshell as presented in textbooks (Intriligator, 1971; Henderson & Quandt, 1980; Varian, 1984).

Such presentation is based on a generic notion of input and output. Without distinguishing primary factors from ingredient inputs, it implicitly assumes that all inputs have a nonnegative marginal product, and that there is a region in the input space in which the Hessian of the production function is negative definite.

Primary factors and ingredient inputs are fundamentally different. The essence of the former consists in the creation of the value-added as the source of income to be shared by those who provided the services, and the latter are acquired from external sources, whose value has already been created by other firms. Ingredient inputs that are procured are specific to each firm and change with the nature of technologies.

We classify inputs into two classes. The first is the class of primary factors of production (labor, capital, land, and managerial talent); we call them inputs of category 1. The second is the class of all ingredient inputs that are procured from external sources (i.e., raw material, intermediate goods, services, etc.); we call them inputs of category 2. We let the primary factors be denoted by an n-dimensional vector v^1 and the ingredient inputs by an m-dimensional vector v^2. Their spaces are the nonnegative orthants of the respective Euclidean spaces, $R_0{}^n$ and $R_0{}^m$.

We ask: What is wrong with the practice of including both classes of inputs into the production function as its independent arguments as in (4) below?

$$x = f(v^1, v^2) \tag{4}$$

The fact is that output cannot be produced without ingredient inputs. Therefore, increasing any of the primary factors of production, without a simultaneous increase in the ingredient inputs, does not yield any additional output. Likewise, increasing the amount of any ingredient input, if not accompanied by an additional use of factor services, does not result in an increased output either. Thus, once the two classes of inputs were treated as independent arguments of the production function, it would be necessary to distinguish the right-hand and left-hand derivatives of the production function since they are not identical. This means that the differential approach to the determination of inputs would lose its operational advantage.

One way to go around the problem is to define the production function as a function that maps a combination of the primary factors (i.e., inputs of category 1) to the maximum producible amount of output from this combination, under the premise that the inputs of category 2 are acquired from external sources. With this understanding, let the acquisition of ingredient inputs be written as a vector function of the primary factor combination; this function shall be called the Z function. Formally, it is a mapping $Z\colon R_0^n \rightarrow R_0^m$.

$$v^2 = Z(v^1) \text{ where } Z(v^1)^T = [Z^1(v^1), Z^2(v^1), \ldots, Z^m(v^1)],\ v_j^2 = Z^j(v^1). \tag{5}$$

If the function is linear, it can be represented by a matrix of $m \times n$ dimensions with constant elements. Such a case will be exceptional since the acquisition of ingredient inputs depends on specific production technologies that are applied, which, in turn, depend on primary factor combinations. Under the specification of the Z-function, the production function, as a mapping from the space of the primary factors to the output space, can be written as a conditional function of v^1 given $Z(v^1)$.

$$x = g(v^1 | Z(v^1)) \equiv f(v^1) \tag{6}$$

Alternatively, in defining a production function, we may start with a primary production function $x = F(v,z)$ (here we let $v = v^1$ and $z = v^2$) defined on the product space $R_0^n \times R_0^m$, regardless of the question of differentiability, as in (4), and derive the production function of the form of $x = f(v)$ as a projection of this function onto the x-v space. That is, given $x = F(v,z)$, find for each v a unique z^* such that

$$F_{z_i}(v,z) = 0 \text{ for all } i, \text{ for } z > z^*, \text{ and}$$
$$F_{z_i}(v,z) > 0 \text{ for all } i, \text{ for } 0 < z < z^*, \tag{7}$$

where $F_{z_i}(v,z) \equiv \frac{\partial F}{\partial z_i}$

Then, for the same v, define the set:

$$\{z\}_v = \{z|F(v,z) \geq F(v,z^*)\}. \tag{8}$$

And, choose the minimal element of this set,

$$z_{\min} = \min\{z\}_v \tag{9}$$

The association of v with this $z_{\min}$ gives the *Z*-function $z = Z(v)$ above. Substituting this function into $x = F(v,z)$ gives

$$x = F(v, Z(v)) \equiv \hat{F}(v) \tag{10}$$

which is the projection of $x = F(v,z)$ onto the *x*-*v* space. This is identical to what function (6) above represents. See Figure 1.

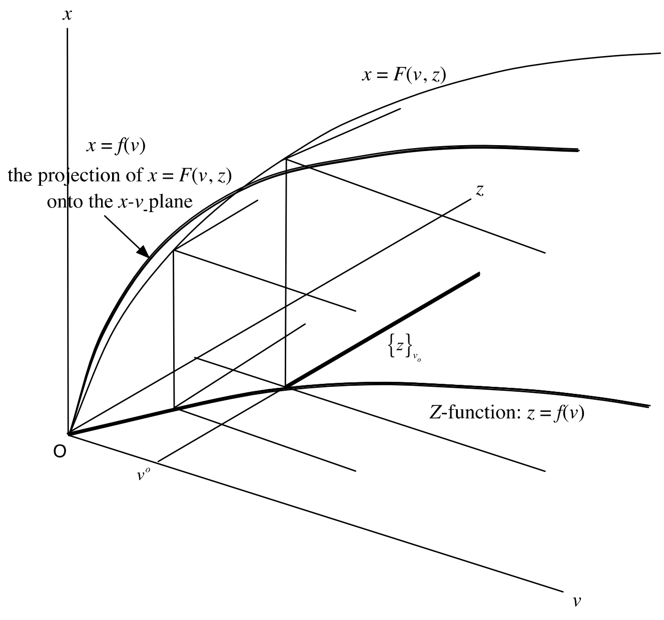

Figure 1. *Production Function*

3. Profit maximization

The production function as a mapping from the space of the primary factors to the output space, mediated by the *Z*-function, was written as $x = f(v^1)$. In the sequel, let it be assumed that the production function and the *Z*-function are twice continuously differentiable. The production function here is stipulated conditional on the acquisition of ingredient inputs, which itself depends on the choice of the primary factor combination. Hence, the firm, in maximizing its profits, acts on this knowledge; that is, how much to employ of factor services and how much to acquire of ingredient inputs are simultaneously determined in profit maximization. Hereafter, we denote the vector of the primary factors by v and the vector of the ingredient inputs by z.

The cost of production, denoted $C(v;w,s,A)$, is specified as

$$C(v;w,s,A) = A + wv + sz = A + \sum_{i=1}^{n} w_i v_i + \sum_{j=1}^{m} s_i z_i \tag{11}$$

where w is a price vector for v and s is a price vector for z, with subscript *i* denoting the *i*-th component; *A* stands for any other cost that is beyond the control of the firm.

The profit function, with the cost of the acquisition of ingredient inputs from external sources taken into account, is defined as

$$\Pi(v;w,s,A) \equiv pf(v) - C(v;w,s,A) \tag{12}$$

The maximization of this profit with respect to the primary factors yields the following first order conditions:

$$pf_i(v) - w_i - \sum_{j=1}^{n} s_j \frac{\partial Z^j(v)}{\partial v_i} = 0,\ i = 1,2,\dots,n, \text{ or, alternatively,}$$

$$p = \frac{w_i + \sum_{j=1}^{m} s_j \dfrac{\partial Z^j(v)}{\partial v_i}}{f_i} = \frac{w_i}{f_i} + \frac{\sum_{j=1}^{m} s_j \dfrac{\partial Z^j(v)}{\partial v_i}}{f_i}, \tag{13}$$

which, together with the marginal cost (74) obtained section 5, implies that the price is equal to the marginal cost of production. The second order condition is the negative definiteness of the Hessian, $H_\Pi(v)$, of $\Pi(v;w,s,A)$, where

$$H_\Pi(v) = pH_f(v) - \sum_{j=1}^{m} s_j H_{Z^j}(v) \tag{14}$$

where $H_\Pi(v) \equiv \left[f_{ij} \right]$ and $H_{Z^k}(v) \equiv \left[Z^k_{ij} \right]$ $\left(\text{the Hessian of } Z^k(v)\right)$.

Alternatively, we may define the value-added function and characterize the optimality conditions in terms of this function.[4] Define the value-added function as

$$V(v;p,s) \equiv pf(v) - \sum_{j=1}^{m} s_j Z_i^j(v). \tag{15}$$

The first order conditions are given as

$$V_i(v;p,s) = w_i,\ i = 1,2,\dots,n, \tag{16}$$

where $V_i(v;p,s) = pf_i(v) - \sum_{j=1}^{m} s_j \dfrac{\partial Z_i^j(v)}{\partial v_i}$.

The term $V_i(v;p,s)$ shall be designated here *the marginal value-added product of primary factor i*; it represents an incremental value-added created by an extra unit of factor service *i*. Condition (16) says that this marginal value-added product is equal to the factor price. Likewise, the second order condition is stated in terms of the negative definiteness of the Hessian, $H_V(v)$, of the value-added function $V(v;p,s)$, which is identical to the Hessian of the profit function (14), i.e.,

$$H_V(v) = pH_f(v) - \sum_{j=1}^{m} s_j H_{Z^j}(v) = H_\Pi(v). \tag{17}$$

There are n first-order conditions for the primary factors, from which, v^1 can be solved as functions of $p, w,$ and s, provided that the Jacobian condition is satisfied (which is met by the second-order condition).

Several observations are worth making: First, the first order condition (13) requires that the marginal value product of a primary factor be equal to the sum of its price and the induced change in the acquisition cost of ingredient inputs, i.e.,

$$pf_i(v) = w_i + \sum_{j=1}^{m} s_j \frac{\partial Z_i^j(v)}{\partial v_i}, \tag{18}$$

or, equivalently, that the marginal value-added product of a primary factor be equal to its price, as expressed in condition (16).

Compare (18) with the traditional condition:

$$pf_i(v) = w_i. \tag{19}$$

Because the sum of the induced changes $\sum_j s_j[\partial Z^j(v)/\partial v_i]$ is positive, condition (18) implies that the employment level of factor i is less than the level implied by (19). It is the marginal value-added product of labor, not the marginal product of labor, that is equated to the wage rate. Likewise, it is the marginal value-added product of capital, not the marginal product of capital, that is equated to the rental rate of capital. Figure 2 shows the difference between (18) and (19) for the case of labor. The marginal value-added product of labor curve lies below the marginal value product of labor curve. If there are technological innovations in the production of ingredient inputs so that the term $\sum_j s_j[\partial Z^j(v)/\partial v_i]$ falls, the marginal value-added product of labor shifts upward, which allows the firm to hire more workers. This explains why widespread innovations at various levels of value chains can have a significant impact on the employment level, and why firms constantly search for the most efficient (cost-saving) producers of intermediate goods. It also explains why industrialization innovations that raise the returns to scale through value chains is strategically complementary; that is, adoption of such innovations at many divided tasks makes it profitable for firms performing the remaining tasks to be equally innovative.

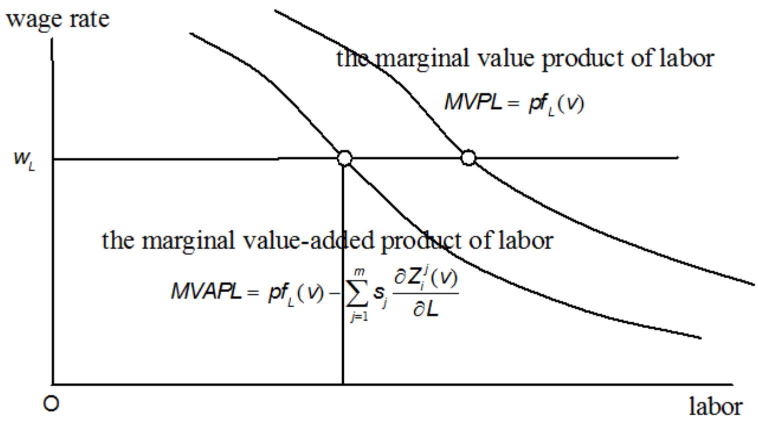

Figure 2. *The marginal value-added product of labor*

Second, because profits are now defined as the difference between the value-added created and the factor payments (not the difference between the total revenue and the factor payments), the second order condition needs to be stated in terms of the Hessian of the value-added function. The negative definiteness of the Hessian of the production function is not enough, unless the *Z*-function is linear, in which case the Hessian of the value-added function coincides with that of the production function.

Third, the homogeneity of the conventional factor demand and output supply functions must be reinterpreted. To see this point, solve (18) for v_i, $i = 1,2,...,n$, each as a function of p, w, and s, and write these functions as

$$v_i = h^i(p,w,s),\ i = 1,2,...,n. \tag{20}$$

With these solutions inserted into the production function $x = f(v)$, the output supply is ob- tained as a function of *p*, *w*, and *s*.

$$x = f\left(h^1(p,w,s), h^2(p,w,s),..., h^n(p,w,s)\right) \equiv F(p,w,s) \tag{21}$$

It is evident from (18) that the factor demand functions, $h^i(p,w,s)$, are homogeneous of degree zero in p, w, and s, but not in *p* and *w* as in the conventional theory of the firm. It follows from this homogeneity that the output supply function $F(p,w,s)$ is equally homogeneous of degree zero in *p*, *w*, and *s*, but not in *p* and *w*.

Fourth, if both the production function $x = f(v)$ and the input requirement functions, $Z^j(v)$, $j = 1,2,...,m$, are homogeneous of degree one in v, so is the value-added function in v. In this case, by Euler's theorem on homogeneous functions, it holds that

$$V(v;p,s) = \sum_{i=1}^{n}\left[pf_i - \sum_{j=1}^{m} s_j Z_i^j(v)\right]v_i. \tag{22}$$

Substituting (16) into (22) then gives

$$V(v;p,s) = \sum_{i=1}^{n} w_i v_i. \tag{23}$$

That is, all of the value-added created is fully distributed to the primary factors of production. To the extent that the ingredient input requirement function $Z(v)$ is non-linear, such homogeneity does not hold in general, which implies that even if the production function is linear homogeneous, there may still be a residual after income is distributed between labor and capital. This may have something to do with a residual term in the growth accounting. We will touch on this point later. One more point: It is evident from (15) that the value-added function is linear homogeneous in p and s.

4. Comparative Statics of Profit Maximization

Assume that the solution is interior. With the input requirement function $Z(v)$, the problem of profit maximization can be expressed as:

$$\underset{x,v}{Max}\,\Pi(x,v;p,w,s)=px-\left[A+\sum_{j=1}^{n}w_iv_i+\sum_{j=1}^{m}s_jZ^j(v)\right] \tag{24}$$

subject to: $g(x,v)\equiv f(v)-x=0$.

Write the Lagrangian $L(x,v,\lambda:p,w,x)$ as

$$L(x,v,\lambda;p,w,x)\equiv px-\left[A+\sum_{j=1}^{n}w_iv_i+\sum_{j=1}^{m}s_jZ^j(v)\right]+\lambda[f(v)-x]. \tag{25}$$

The first order conditions are:

$$\frac{\partial L}{\partial x}=p-\lambda=0, \tag{26}$$

$$\frac{\partial L}{\partial v_i}=-w_i-\sum_{j=1}^{m}s_j\frac{\partial Z^j(v)}{\partial v_i}+\lambda f_i=0 \tag{27}$$

$$x-f(v)=0. \tag{28}$$

Combining (26) and (27) and using the value-added function $V(v;p,s)$ gives

$$V_i(v)=w_i,\ i=1,2,\dots,n, \tag{29}$$

where $V_i(v)\equiv pf_i(v)-\sum_{i=1}^{m}s_j\dfrac{\partial Z^j(v)}{\partial v_i}$.

The second order condition is stated in terms of the negative definiteness of the Hessian of the Lagrangian function, i.e., $\xi H_L(x,v)\xi'<0$ for all $\xi\neq 0$ such that $[-1\ \ f_1\ \ f_2\ \dots\ f_n]\xi'=0$, which is equivalent to the following bordered Hessian condition:

$$(-1)^r\begin{bmatrix} 0 & 1 & -f_1 & -f_2 & . & . & . & -f_n \\ 1 & 0 & 0 & 0 & . & . & . & 0 \\ -f_1 & 0 & & & & & & \\ -f_2 & 0 & & & & & & \\ . & 0 & & pH_f(v) & - & \sum_{j=1}^{m}s_jH_{Z^j}(v) & & \\ . & 0 & & & & & & \\ . & 0 & & & & & & \\ -f_n & 0 & & & & & & \end{bmatrix}>0,\ r=2,3,\dots,n \tag{30}$$

The fundamental equations of comparative statics are obtained from (26), (27), and (28) as:

$$\begin{bmatrix} 1 & -f_1 & \ldots & -f_n \\ 0 & V_{11} & \ldots & V_{1n} \\ . & . & . & . \\ -f_n & V_{n1} & \ldots & V_{nn} \end{bmatrix} \begin{bmatrix} dx \\ dv_1 \\ . \\ dv_n \end{bmatrix} = \begin{bmatrix} 0 \\ dw_1 - f_1 dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_1} ds_j \\ . \\ dw_n - f_n dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_n} ds_j \end{bmatrix} \tag{31}$$

Assuming that the coefficient matrix is nonsingular, $(dx, dv_1, \ldots, dv_n)$ can be solved as

$$\begin{bmatrix} dx \\ dv_1 \\ . \\ dv_n \end{bmatrix} = \begin{bmatrix} 1 & -f_1 & \ldots & -f_n \\ 0 & V_{11} & \ldots & V_{1n} \\ . & . & . & . \\ -f_n & V_{n1} & \ldots & V_{nn} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ dw_1 - f_1 dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_1} ds_j \\ . \\ dw_n - f_n dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_n} ds_j \end{bmatrix}$$

$$= \begin{bmatrix} 1 & [f, f_2, \ldots, f_n] H_V(v)^{-1} \\ 0 & \\ . & H_V(v)^{-1} \\ 0 & \end{bmatrix} \begin{bmatrix} 0 \\ dw_1 - f_1 dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_1} ds_j \\ . \\ dw_n - f_n dp + \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_n} ds_j \end{bmatrix}, \tag{32}$$

or, more specifically, as

$$dx = [f_1, f_2, \ldots, f_n][H_V(v)^{-1}] \left[\begin{bmatrix} dw_1 \\ dw_2 \\ . \\ dw_n \end{bmatrix} - \begin{bmatrix} f_1 dp \\ f_2 dp \\ . \\ f_n p \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_1} ds_j \\ \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_2} ds_j \\ . \\ \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_n} ds_j \end{bmatrix} \right] \tag{33}$$

$$\begin{bmatrix} dv_1 \\ dv_2 \\ . \\ dv_n \end{bmatrix} = [H_V(v)^{-1}] \left[\begin{bmatrix} dw_1 \\ dw_2 \\ . \\ dw_n \end{bmatrix} - \begin{bmatrix} f_1 dp \\ f_2 dp \\ . \\ f_n p \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_1} ds_j \\ \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_2} ds_j \\ . \\ \sum_{j=1}^{m} \frac{\partial Z^j}{\partial v_n} ds_j \end{bmatrix} \right] \tag{34}$$

These equations yield the following comparative statical results. First, the slope of the output supply function in the *p-x* space is positive. That is, because the matrix $\left[H_V(v)^{-1}\right]$ is negative definite by the second order condition, it follows that

$$\frac{\partial x}{\partial p} = -\left[\frac{\partial f}{\partial v}\right]\left[H_V(v)^{-1}\right]\left[\frac{\partial f}{\partial v}\right]' > 0 \text{ where } \left[\frac{\partial f}{\partial v}\right] \equiv [f_1, f_2, ..., f_n]. \tag{35}$$

Second, the effects of the changes in the prices of the primary factors on the supply of output x are solved as

$$\left[\frac{\partial x}{\partial w_1}, \frac{\partial x}{\partial w_2}, ..., \frac{\partial x}{\partial w_n}\right] = \left[f_1, f_2, ..., f_n\right]\left[H_V(v)^{-1}\right] \tag{36}$$

Third, the effects of the changes in the prices of ingredient inputs on the supply of output are solved as

$$\frac{\partial x}{\partial s_j} = \left[f_1, f_2, ..., f_n\right]\left[H_V(v)^{-1}\right]\begin{bmatrix} \frac{\partial Z^j(v)}{\partial v_1} \\ \frac{\partial Z^j(v)}{\partial v_2} \\ . \\ \frac{\partial Z^j(v)}{\partial v_n} \end{bmatrix}, j = 1, 2, ..., m. \tag{37}$$

While the sign of $\partial x / \partial s_j$ is generally ambiguous *a priori*, it is negative in the case in which $\left[\frac{\partial Z^j}{\partial v_1}, \frac{\partial Z^j}{\partial v_2}, ..., \frac{\partial Z^j}{\partial v_n}\right]$ is a scalar multiple of $\left[f_1, f_2, ..., f_n\right]$. That is, if it holds that

$$\left[\frac{\partial Z^j}{\partial v_1}, \frac{\partial Z^j}{\partial v_2}, ..., \frac{\partial Z^j}{\partial v_n}\right] = \alpha\left[f_2, f_2, ..., f_n\right] \tag{38}$$

where α is a scalar, then, by the negative definiteness of $\left[H_V(v)^{-1}\right]$, all of the effects in (36) are negative; i.e.,

$$\frac{\partial x}{\partial s_j} = \alpha\left[f_2, f_2, ..., f_n\right]\left[H_V(v)^{-1}\right]\begin{bmatrix} f_1 \\ f_2 \\ . \\ f_n \end{bmatrix} < 0, j = 1, 2, ..., m. \tag{39}$$

Finally, the effects of the changes in the prices of the ingredient inputs on the amount employed of the primary factors are obtained as

$$\begin{bmatrix} \frac{\partial v_1}{\partial s_j} \\ \frac{\partial v_2}{\partial s_j} \\ . \\ \frac{\partial v_n}{\partial s_j} \end{bmatrix} = \left[H_V(v)^{-1}\right]\begin{bmatrix} \frac{\partial Z^j}{\partial v_1} \\ \frac{\partial Z^j}{\partial v_2} \\ . \\ \frac{\partial Z^j}{\partial v_n} \end{bmatrix}, j = 1, 2, ..., m. \tag{40}$$

5. Comparative statics of cost minimization

We proceed to the problem of cost minimization. This problem is formulated as:

$$\underset{v}{Minimize}\, C(v; w, s, A) \equiv A + \sum_{j=1}^{n} w_i v_i + \sum_{j=1}^{m} s_j Z^j(v) \tag{41}$$

subject to: $f(v) = \bar{x}$ ($\bar{x}$ given).

With the Lagrangian $L(v, \lambda; w, s, A)$ written as

$$L(v, \lambda; w, s, A) \equiv A + \sum_{j=1}^{n} w_i v_i + \sum_{j=1}^{m} s_j Z^j(v) + \lambda[\bar{x} - f(v)], \tag{42}$$

the first order conditions are obtained as:

$$\frac{\partial L}{\partial v_i} = w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i} - \lambda f_i = 0 \tag{43}$$

$$\frac{\partial L}{\partial \lambda} = \bar{x} - f(v) = 0 \tag{44}$$

The second order condition is that the Hessian of the Lagrangian function $L(v, \lambda; w, s, A)$ is positive definite for all non-zero vectors $\xi \neq 0$ such that $[-f_1, -f_2, ..., -f_n]\xi' = 0$, which is equivalent to the bordered Hessian condition that $n-1$ principal minors of the following matrix are negative.

$$\begin{bmatrix} & & & -f_1 \\ \sum_{j=1}^{m} s_j H_{Z^j}(v) & - & \lambda H_f(v) & -f_2 \\ & & & . \\ & & & -f_n \\ -f_1 & -f_2 & . & -f_n \quad 0 \end{bmatrix} \tag{45}$$

With the Jacobian condition satisfied (which is assured by the second order condition), the first order conditions (43) and (44) can be solved for v_i and λ:

$$v_i = H^i(x; w, s),\ i = 1, 2, ..., n \tag{46}$$

$$\lambda = J(x; w, s) \tag{47}$$

We now examine the properties of these functions by making use of the fundamental equations of comparative statics:

$$\begin{bmatrix} & & & -f_1 \\ \sum_{j=1}^{m} s_j H_{Z^j}(v) & - & \lambda H_f(v) & -f_2 \\ & & & . \\ & & & -f_n \\ -f_1 & -f_2 & . & -f_n \quad 0 \end{bmatrix} \begin{bmatrix} dv_1 \\ dv_1 \\ . \\ . \\ dv_n \\ d\lambda \end{bmatrix} = \begin{bmatrix} -dw_1 - \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_1} ds_j \\ . \\ -dw_n - \sum_{j=1}^{m} \frac{\partial Z^j(v)}{\partial v_n} ds_j \\ -dx \end{bmatrix} \tag{48}$$

Solving for dv_k and $d\lambda$ gives:

$$dv_k = \frac{1}{D}\left[\sum_{i=1}^{n}\left\{-dw_i - \sum_{j=1}^{m}\frac{\partial Z^j(v)}{\partial v_i}ds_j\right\}D_{ik} - D_{n+1,k}dx\right] \tag{49}$$

$$d\lambda = \frac{1}{D}\left[\sum_{i=1}^{n}\left\{-dw_i - \sum_{j=1}^{m}\frac{\partial Z^j(v)}{\partial v_i}ds_j\right\}D_{i,n+1} - D_{n+1,n+1}dx\right] \tag{50}$$

where D is the determinant of the coefficient matrix and D_{ij} denotes its cofactor. These yield the following comparative statical information.

$$\frac{\partial v_k}{\partial w_j} = -\frac{D_{jk}}{D} \tag{51}$$

$$\frac{\partial v_k}{\partial w_k} = -\frac{D_{kk}}{D} < 0 \text{ (by the bordered Hessian condition)} \tag{52}$$

$$\frac{\partial v_k}{\partial s_j} = -\sum_{i=1}^{n}\frac{\partial Z^j(v)}{\partial v_i}\frac{D_{ik}}{D} \tag{53}$$

$$\frac{\partial v_k}{\partial x} = -\frac{D_{n+1,k}}{D} \tag{54}$$

$$\frac{\partial \lambda}{\partial w_j} = -\frac{D_{j,n+1}}{D} \tag{55}$$

$$\frac{\partial \lambda}{\partial s_j} = -\sum_{i=1}^{n}\frac{\partial Z^j(v)}{\partial v_i}\frac{D_{i,n+1}}{D} \tag{56}$$

$$\frac{\partial \lambda}{\partial x} = -\frac{D_{n+1,n+1}}{D} \tag{57}$$

Several observations are made. First, the Lagrangian multiplier λ can still be interpreted as the marginal cost, where the cost of production, with solution (46) inserted, is written as

$$\hat{C}(x;w,s,A) \equiv C\left(H^i(x;w,s);w.s,A\right)$$
$$= A + \sum_{j=1}^{n} w_i H^i(x;w,s) + \sum_{j=1}^{m} s_j Z^j\left(H^i(x;w,s)\right). \tag{58}$$

This can be demonstrated by taking the derivative of (58) and substituting (43) and (54) therein:

$$\frac{\partial \hat{C}}{\partial x} = \sum_{i=1}^{n} w_i \frac{\partial H^i(x;w,s)}{\partial x} + \sum_{j=1}^{m} s_j \sum_{i=1}^{n}\frac{\partial Z^j(v)}{\partial v_i}\frac{\partial H^i(x;w,s)}{\partial x}$$
$$= \sum_{i=1}^{n}\left(-\sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i} + \lambda f_i\right)\left(-\frac{D_{n+1,i}}{D}\right) + \sum_{j=1}^{m} s_j \sum_{i=1}^{n}\frac{\partial Z^j(v)}{\partial v_i}\left(-\frac{D_{n+1,i}}{D}\right)$$
$$= \lambda\sum_{i=1}^{n}(-f_i)\left(\frac{D_{n+1,i}}{D}\right) = \lambda\frac{D}{D} = \lambda \tag{59}$$

Combining (59) with (43) gives

$$\frac{\partial \hat{C}}{\partial x} = \lambda = \frac{w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i}}{f_i}, \tag{60}$$

which shows that

$$\frac{\partial \hat{C}}{\partial x} \neq \frac{w_i}{f_i}. \tag{61}$$

That is, the marginal cost of production is no longer equal to the factor price divided by its marginal product.

Second, because of the additional term $\sum_j s_j Z^j(v)$, the iso-cost curves are not linear in general. To see this point, take an iso-cost surface for a fixed level of cost, C^o.

$$C(v;w,s,A) \equiv A + \sum_{j=1}^{n} w_i v_i + \sum_{j=1}^{m} s_j Z^j(v) = C^o, \tag{62}$$

which gives

$$\sum_{j=1}^{n} \left(w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i} \right) dv_i = 0. \tag{63}$$

Hence, the slope in the iso-cost surface in the v_i-v_k space is given by

$$\frac{dv_i}{dv_k} = \frac{w_k + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_k}}{w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i}}. \tag{64}$$

This shows that the iso-cost surface is no longer linear unless the term $\sum_j s_j Z^j(v)$ is linear. Note that the first order condition (43) shows that the isoquant surface is tangent to the iso-cost surface. That is,

$$\left[\frac{dv_i}{dv_k}\right]_{f(v)=x} = \frac{w_k + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_k}}{w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(v)}{\partial v_i}} = \left[\frac{dv_i}{dv_k}\right]_{C=C^o} \tag{65}$$

Third, it can be demonstrated that Shephard's lemma holds. That is, given the cost function as a function of the quantity of output with the primary factor prices and the prices of the acquired ingredient inputs as its parameters, the primary factor demand functions can be obtained by differentiating the cost function partially with respect to the primary factor prices. This can be demonstrated as follows: Recall the cost function (58), which was written as

$$\hat{C}(x;w,s,A) = A + \sum_{j=1}^{n} w_i H^i(x;w,s) + \sum_{j=1}^{m} s_j Z^j\left(H^i(x;w,s)\right). \tag{66}$$

Differentiating this cost function partially with respect to w_i yields:

$$\frac{\partial \hat{C}(x;w,s,A)}{\partial w_i} = H^i(x,w,s) + \sum_{k=1}^{n} w_k \frac{\partial H^k(x;w,s)}{\partial w_i} + \sum_{j=1}^{m}\sum_{k=1}^{n} s_j \frac{\partial Z^j}{\partial v_k}\frac{\partial H^k(x;w,s)}{\partial w_i}$$
$$= H^i(x,w,s) + \sum_{k=1}^{n}\left[w_k + \sum_{j=1}^{m} s_j \frac{\partial Z^j}{\partial v_k}\right]\frac{\partial H^k(x;w,s)}{\partial w_i}. \tag{67}$$

Differentiating the production function

$$x = f\left(H^1(x;w,s), H^2(x;w,s), ..., H^n(x;w,s)\right) \tag{68}$$

with respect to w_i and making use of the first order condition (43) gives

$$\sum_{k=1}^{n} \frac{\partial f}{\partial v_k}\frac{\partial H^k}{\partial w_i} = \sum_{k=1}^{n}\frac{1}{\lambda}\left[w_k + \sum_{j=1}^{m} s_j \frac{\partial Z^j}{\partial v_k}\right]\frac{\partial H^k(x;w,s)}{\partial w_i} = 0. \tag{69}$$

Substituting (69) into (67) gives the Shephard's lemma.

$$\frac{\partial \hat{C}(x;w,s,A)}{\partial w_i} = H^i(x,w,s) \tag{70}$$

Fourth, the concavity of the cost function (58) holds in (w,s). To see it, take two price vectors, (w^1,s^1) and (w^2,s^2), and denote the minimum cost combinations of the primary factors under these prices by $\hat{v}^1$ and $\hat{v}^2$, respectively. And, consider the following convex combinations:

$$w^* = \theta w^1 + (1-\theta)w^2 \text{ and } s^* = \theta s^1 + (1-\theta)s^2,\ 0<\theta<1, \tag{71}$$

and denote the minimum cost combination of the primary factors under (w^*,s^*) by v^*. Then it follows:

$$\begin{aligned}
&\hat{C}(v^*;w^*,s^*,A) = A + w^* v^* + s^* Z(v^*)\\
&= A + \left[\theta w^1 + (1-\theta)w^2\right]v^* + \left[\theta s^1 + (1-\theta)s^2\right]Z(v^*)\\
&= \theta\left[A + w^1 v^* + s^1 Z(v^*)\right] + (1-\theta)\left[A + w^2 v^* + s^2 Z(v^*)\right]\\
&\geq \theta\left[A + w^1 \hat{v}^1 + s^1 Z(\hat{v}^1)\right] + (1-\theta)\left[A + w^2 \hat{v}^2 + s^2 Z(\hat{v}^2)\right]\\
&= \theta\hat{C}(x;\ w^1,\ s^1,A) + (1-\theta)\hat{C}(x;\ w^2,\ s^2,A).
\end{aligned} \tag{72}$$

This demonstrates that the cost function is concave in (w,s).

Finally, with (46) taken into account, the profit function can be expressed as a function of output x.

$$\hat{\Pi}(x;w,s) = px - \left(A + \sum_{i=1}^{n} w_i H^i(x;w,s) + \sum_{j=1}^{m} Z^j\left(H^i(x;w,s)\right)\right) \tag{73}$$

Maximizing this profit with respect to x gives

$$p = MC \tag{74}$$

where the marginal cost MC is given by

$$MC \equiv \frac{\partial \hat{C}(x; w.s.A)}{\partial x} = \sum_{i=1}^{n} w_i \frac{\partial H^i(x; w, s)}{\partial x} + \sum_{i=1}^{n}\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}\frac{\partial H^i(x; w, s)}{\partial x}.$$

We also know from (13) that

$$pf_i = w_i + \sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i} \text{ or } p = \frac{w_i}{f_i} + \frac{\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}}{f_i}. \tag{75}$$

So, the price equals the sum of two terms: The first term w_i / f_i represents the incremental cost of factor services required to produce an extra unit, and the second term represents the incremental cost of ingredient inputs required for this production. The price is no longer equals w_i / f_i. This is reconfirmed by combining (74) and (75), which establishes an identity:

$$\frac{w_i}{f_i} + \frac{\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}}{f_i} = \sum_{i=1}^{n} w_i \frac{\partial H^i(x; w, s)}{\partial x} + \sum_{i=1}^{n}\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}\frac{\partial H^i(x; w, s)}{\partial x} \tag{76}$$

w_i / f_i amounts to $w_i \frac{\partial H^i(x; w, s)}{\partial x}$, and $\left(\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}\right) / f_i$ to $\sum_{i=1}^{n}\sum_{j=1}^{m} s_j \frac{\partial Z^j(x; w, s)}{\partial v_i}\frac{\partial H^i(x; w, s)}{\partial x}$.

6. GDP and Ingredient Inputs Acquired from Abroad

The aggregate production function, in value-added terms, has been estimated as a function of labor and capital. The Cobb-Douglas production function (Cobb & Douglas, 1928; Douglas, 1976) has played a particularly important role in such estimation, and the obtained function has been used to estimate the magnitude of the residual in the growth accounting that is attributable to technological innovations as exemplified in Solow (1957) or to capture the stochastic process of such innovations that is mimicked in real business cycle models. Such estimation subsumes all of the effects of the ingredient inputs acquired from external sources, domestic or foreign, in the total value-added created. As a matter of fact, many national economies are exposed to a variety of real shocks that are reflected in sharp rises in the prices of the critical materials or products (e.g., oil and natural gas, rare earth metals). Such price rises hinder the creation of the value-added in the domestic economy and reduces the employment of factor services and factor payments. The oil shocks in the 70s caused a serious contraction of this nature. The effects of such shocks were compounded by the demand externalities that spilled over across many national economies at the time.

One way to capture the effects of such shocks on the production function that is based on the value-added is to modify it by a multiplicative factor that depends on the acquisition of the crucial inputs from foreign sources as in

$$Q = \delta(E^1, E^2, ..., E^l)F(K, L) \quad (77)$$

where $(E^1, E^2, ..., E^l)$ is a list of such inputs and $\delta(E^1, E^2, ..., E^l)$ captures the effect on the production. At the time of oil shock, $Q = \delta(E)F(K, L)$ (E is the quantity of oil at disposal) would have captured the effect of the shock, where $F(K, L)$ represents what the country would be able to produce under a normal supply of oil. When actual oil shocks hit, the oil supply itself was severely restrained worldwide. Hence, the impact of such shocks on the domestic product, through an unusually large payment for the imported oil, would not still fully explain a significant reduction in the recorded product, the reason being that even if high prices are paid, the quantity imported may be rationed, which restricts the production of final goods through the quantity constraint. If this is the case, there should be another way of capturing such effects with an explicit accounting of the acquisition of such inputs at the level of individual firms. In general, the economy as a whole imports crucial materials, energy goods, and many other intermediate goods from the competitive suppliers globally. Such accounting informs that the so-called Solow residual reflects not only pure technological changes that shift the production function domestically but also real shocks that slow down or facilitate the creation of the value-added that is mediated by the acquisition of intermediate goods from external sources. If this creation is hampered by a supply shock of intermediate goods, the Solow residual will record a reduction. On the other hand, if the same creation is facilitated by innovations in the supply of intermediate goods that occur in other countries (with a consequent fall in the supply prices of intermediate goods in the world market), the Solow residual will record a gain. Thus, the Solow residual is a mixture of various impacts. This is one important reason that lies behind the current research on the total factor productivity as affected by intermediate goods. It would be informative to consider the acquisition of such goods more explicitly in accounting for the fluctuations of the domestic product. The fact that the globalized economy is functionally and spatially fragmented through division of production activities can also account for linkages that transmit shocks across vertically aligned firms in value chains through a multiplier process.

7. Conclusion

Firms use two types of inputs: the primary factors and ingredient inputs. The traditional theory is focused exclusively on primary factors. For this reason, the profit maximization is expressed in terms of the marginal value product of primary factors or in terms of the marginal cost that arises solely from an incremental employment of such factors. This way of describing the profit maximizing behavior is misleading in light of the fact that it is the value-added that firms create by employing the primary factors. If so, the profit maximization principle should be expressed more accurately in terms of the marginal value-added product of primary factors, or in terms of the marginal cost that includes not only the cost of additional factor services required to produce an extra unit but also the cost of the ingredient inputs required for this production. With this insight, this paper attempted to analyze the profit maximizing and the cost minimizing behavior of firms by distinguishing primary factors and ingredient inputs that are procured externally. The production function was defined as a mapping from the space of primary factors to the output space while necessary ingredient inputs are acquired from external sources in accordance with the input requirement functions; the value-added function was defined as the difference between the market value of the output produced and the cost of the ingredient inputs procured externally; the profit function was defined as the difference between this value-added function and the factor payments; and the cost function was defined as the minimum cost to produce a given quantity of output when the cost of the acquisition of the ingredient inputs is added to the cost of factor payments.

Specifically, the first-order condition of profit maximization was shown to require that the marginal value-added product of a primary factor be equal to its price, and the second-order condition was stated in terms of the negative definiteness of the Hessian of the value-added function rather than in terms of the negative definiteness of the Hessian of the production function. Cost minimization, on the other hand, was shown to require that the marginal cost of production be equal to the sum of an incremental cost of the factor payment and an incremental cost of acquiring the necessary ingredient inputs. Observations were made with respect to the properties of the cost function, the factor demand and output supply functions, and the value-added function. In particular, the value-added function is linear homogeneous only if the production function and the input requirement functions are both linear homogeneous. Shephard's lemma holds on the cost function that includes the cost of acquiring ingredient inputs. And, the cost function is concave with respect to the prices of ingredient inputs and primary factors. The comparative statics of profit maximization revealed that the optimum output and factor employment respond to changes in the prices of ingredient inputs as well as to the prices of primary factors. The comparative statics of cost minimization equally demonstrated that the minimum cost of producing a given quantity is affected by the prices of ingredient inputs as well as by the prices of primary factors. The highlight of the paper is that it is the concept of the marginal value-added product of primary factors that characterizes the profit maximizing behavior of firms, in contrast to the marginal value product of such factors. The paper has clarified the ambiguity that surrounds the question on how to characterize the behavior of firms and the cost of production when the procurement of ingredient inputs from external sources is fully taken into account along with primary factors.

Notes

[1] A distinction among different types of inputs has been made in the literature. For instance, in measuring real net output or in constructing an index of such output either for a given industry or for an entire economy, inputs were differentiated by whether they are primary factors of production such as labor and capital or purchased inputs from other industries (David, 1962, 1966); Sims, 1969). A similar distinction between primary factors of production and intermediate inputs was addressed by Bruno (1978), Khang (1971), and Diewert (1978). But, in their treatment, intermediate inputs are entered into the production function in the same way as the primary factors.

[2] For instance, Ferguson (1969, p. 71) defines a production function as follows:

… a production function shows the maximum output attainable from any specified set of inputs, i.e., any set of quantities of ingredient inputs and flows of services of other inputs. In general, no further limitations are imposed except that the set of outputs and inputs must be nonnegative. Finally, the production function is a single valued mapping from input space into output space inasmuch as the maximum attainable output for any stipulated set of inputs is unique.

[3] Khang (1971) and Bruno (1978), distinguishing intermediate inputs from primary factors of production, considered value-added functions in place of restricted profit functions in establishing the duality between production structures and value-added functions. In their approaches, however, primary factors of production and intermediate inputs are both entered into a production function, and this function is assumed twice continuously differentiable with positive partial derivatives and with a negative definite Hessian matrix. Their analysis, therefore, raises a fundamental question as to why a production function can still be assumed to satisfy these properties when intermediate inputs, which are necessary for production, are distinctly differentiated from primary factors of production. It is this question that is addressed in this paper.

[4] The definition of a value-added function that is adopted in this paper is different from that of Khang (1971), Bruno (1978), and Diewert (1978). The latter is basically in the form of a variable profit function, which gives the maximum value-added that can be produced from a given combination of primary factors of production when output and intermediate goods that are technically feasible are varied with their prices being given. For example, Bruno (1978, p. 5) defines a (nominal) value-added function by

$$G(L,\pi,P)=\underset{X,M}{Max}\{\pi X-P'M\,|X\le X(L,M)\}$$
$$=\pi\hat{X}(L,P,\pi)-P'\hat{M}(L,P,\pi)$$

where $X(L,M)$ is a production function (of a single, composite gross output) with L and M representing primary factors of production and intermediate goods; π and P (vector) are the prices of output and intermediate goods; $\hat{X}(L,P,\pi)$ and $\hat{M}(L,P,\pi)$ are restricted profit maximizing quantities of X and M given as functions of L, P, and π. Apart from normalization, this is essentially what Khang (1971) proposed.

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