Journal of Economics Bibliography
www.kspjournals.org
Volume 3 December 2016 Issue 4

Economic Inequality as a Statistical Outcome

By Oded KAFRI [†] & Eli FISHOF

Abstract. We argue that the economic inequality which is found in the OECD countries and in the salaries of the top executives' in the Fortune 100 companies are merely an equilibrium statistical outcome, similar to that of the energy distribution among photons in a blackbody. When we treat the photons as people, the radiation modes asocial rank and the photons' energy as wealth; we obtain for the energy distribution among photons: the Gini index, the ratio between rich and poor, the relative poverty, the part of the property held by the upper percentiles, the salaries of CEO's, very similar values to the corresponding numbers of the OECD counties and Fortune 100 companies.
Keywords. Economic inequality, Gini index, Relative poverty, Pareto law.
JEL. C62.

1. Introduction

Income distributive justice is a political subjective phrase related to an income distribution rather than to a scientific issue. Most people believe that income inequality should be as small as possible. Nevertheless, it is understood that a certain gap between the rich and the poor is necessary to stimulate competition between individuals. This competition is the invisible hand of any healthy economy. One may ask if there is an optimal inequality. This question is intriguing both from philosophical and practical points of view. Every society has a strong motivation to have a strong competitive economy on one hand and a social just on the other. These two factors are vital to the quality of life of the people. The governments regulate the net income distribution through taxation, and therefore it is of great importance to find if there is a theoretical criterion for an optimal wealth distribution. Moreover, history teaches us that a high income inequality may lead to political protests and even revolutions. In the words of philosopher Plutarch: "An imbalance between rich and poor is the oldest and most fatal ailment of all republics."

The income inequality research which probably started with Pareto golden rule at the end of the 19th century continues to these days (Ball, 2004). The contemporary physical approach to economy is based on statistical mechanics of ideal gas (Maxwell-Boltzmann), where as the distribution of incomeis compared to the distribution of energy-money among the particles-people (Dragulescu, & Petrova, 2000). However this approach that was applied by econophysicists (Ball, 2004) has not yield profound results. It was suggested previously (Kafri, 2014) that economy can be described more accurately as a network in which the money is a transient quantity exchanged between its nodes. In nature energy and transient energy, which is called heat, have different statistics. Energy has Maxwell

† Kafri Nihul Ltd., Tel Aviv 68165 Israel.
☎. +9 725 259 925 01
✉. kafri.entropy@gmail.com

Boltzmann statistics, and heat (i.e. photons, phonons and alike) obeys Planck statistics. In the network economy each node trades with the other nodes by transactions. Each transaction of money is represented by an integer number. The value of a number is the amount of money transferred and the sign of it is its direction. For example, if a transaction is +A it means that the node received A$, and if the transaction is –B its means that the node paid B$. In order to have an economy one need to add to this model a bank. The bank serves as the memory of the network in which all it transactions are registered. In addition of being a memory the bank is also aregulator. For example, the bank may decide that the balance of a given node, namely the sum of all itstransactions at any given time, cannot be negative. However, in order to have trade the bank should allow at least to some nodes to have a negative balance. In this case we say that the node receives a "credit" from the bank. When the bank issues the payment, it is registered as minus in the loaner-node’s account. But, since the loaner pays with the loan to other nodes, and they deposit this money back in the bank, the total balance of the bank remains zero. We see that the bank is not really affected by crediting the nodes. In fact, the bank generated money from nothing by crediting the nodes, and therefore we may conclude that money is not subject to a conservation law.

At a first glance it seems that in this toy model there is no room for recessions, crisis, economic booms and alike. However, the total amount of money, which reflects the sum of all the transactions between the nodes, is not conserved, and therefore it may be changed due to psychological reasons like fear, optimism or even long period of prosperity that is expected to end.When the total amount of transactions reduces, there is an economic recession, and when it increases there isan economicgrowth.

The network economy model enables us to calculate the distribution of money between people exactly as it was done with the distribution of links among nodes (Kafri, 2014) and the distribution of energy among photons. This distribution, which is called Planck Benford's distribution (Kafri, 2016; Kafri, & Kafri, 2013), with accordance to the intuitive description of the network economy above, is also independent of the total amount of the money of the net or in the total amount of energy of the radiating object. That is to say; the ratio between the various income ranks is only a function of the number of the ranks. This is different from the normal distribution of energy between particles in ideal gas which varies with the total amount of energy of the gas.

The Planck-Benford distributionis basically a manipulation of Planck law (Planck, 1901) which describesthe equilibrium energy distribution in a finite number of radiation modes.The distribution of energy in the modes were calculated by maximizing the entropy (ME) of the radiating body (Kafri, 2016) namely,

$$\varepsilon(n) = \ln\frac{1+\frac{1}{n}}{\ln(N+1)} \tag{1}$$

Where N is the number of the modes, which are interpreted here as the chosen number of income ranks (which might be deciles, percentiles, tenth percentiles or any other positive integer), n is a serial number called here the rank number of the nodes where $n = 1,2,\dots N$. Therefore, the people are the nodes in rank n and $\varepsilon(n)$ is their normalized wealth. If $N = 10$ then $\varepsilon(3)$ is the relative income of the third decile.

Eq. (1) was derived from Planck law (Planck, 1901) for photons, namely n =1/(exp($\beta\varepsilon(n)$)-1); n is the occupation number which is the number of photons in a mode (mode is a radiation distinguishable state), β is a parameter related to temperature – which is determined by the total energy of the system, and ε is the energy-wealth of the photons. If we write Planck's equation differently, namely,

$\varepsilon(n) = \beta^{-1}\ln(1+\frac{1}{n})$, and considering that the normalization factor,

$\sum_{n=1}^{N}\varepsilon(n) = \beta^{-1}\ln(N+1)$, then $\varepsilon(n)/\sum_{n=1}^{N}\varepsilon(n)$ is the relative wealth distribution as expressed by Eq. (1). It is seen that smaller the rank number richer the people in it. Therefore when a number of people are divided randomly in N distinguishable groups, their wealth will decrease with the social rank n, according to Eq. (1).

2. Gini Index

Gini Index is the standard measure of income inequality for countries. It is a single number (ranges from 0 to 1) that is obtained from the relative net income distribution function $\varepsilon(n, N)$. If the income of x percent of the population is $\varepsilon(x)$, then one defines the Lagrange function as $L(x) = \int_0^x \varepsilon(x')dx'$.

Namely, $L(x)$ is the total income of all the population up to the fraction x. If the income is distributed equally, then $\varepsilon(x)$ is constant and $L(x) = x$.

Gini index is defined as $G = \int_0^1 [x - L(x)]\, dx$. If $\varepsilon(x)$ is constant then G is zero. Here we use a discrete version of the Gini index. We divide the population to 10 deciles according to the decreasing n, namely according to increasing income. We designate the fraction of the net income of the n decile by $\varepsilon(n)$ and the discrete Gini index is defined as

$$G = \sum_{i=1}^{10}\sum_{n=1}^{i}[\frac{n}{10} - \varepsilon(11-n)] \qquad (2)$$

$L(11-n)$is the discrete Lorentz curve, namely the fraction of the net income of all the deciles up to the $11-n$ decile, namely,

$L(i) = \sum_{n=1}^{i}\varepsilon(11-n)$ because ε is normalized $L(1) = 1$.

Now we calculate the Gini index for the Planck Benford's distribution of wealth in 10 ranks. Each rank represents a decile of the population having similar income.

In Fig. 1 we see the result of the substitution of Eq. (1) in Eq. (2)

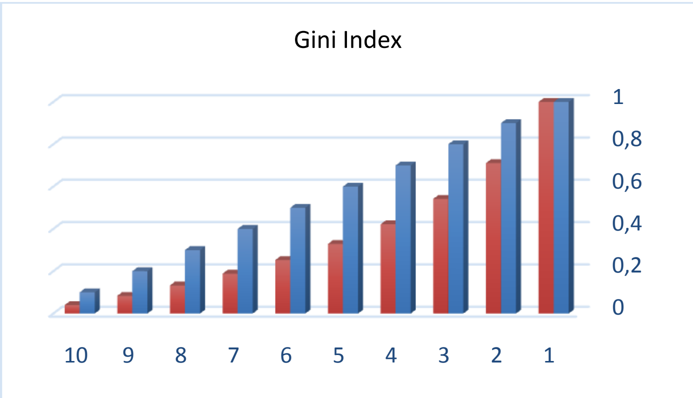

Figure 1. *The blue bars are the accumulated income of the deciles for the case where each decile has the same income. The orang bars are the accumulated income of the deciles with the distribution of Planck Benford. The sum of the differences between the blue bars and the orang bars is the Gini Index.*

This calculation yields G=0.327. It is quiet surprising that the average Gini index of the 35 countries of the OECD in 2012 is almost identical to that obtained here theoretically for network economy in equilibrium, namely G=0.32. Moreover, it is counterintuitive to think that in the free world the highly regulated income inequality will be similar to that of energy inequality among photons. The reason for the surprise is the influence of the governments on Gini index by taxation in order to increase equality and decrease Gini index. Most countries in the world also compensate poor people by supplementary income in addition to taxation. Yet the Gini index is almost identical.

3. The ratio between the incomes of the upper decile and the lowest decile

From Eq. 1 we calculated table 1 that present the relative wealth of the deciles. The richest decile $n = 1$ has 0.289 of the total wealth of the group, and the ratio between the highest income decile and the poorest, according to table 1, is 7.25. The average of the OECD for this ratio is 9.6, which is 32% higher than that of equilibrium countries. This point will be discussed later.

Table 1. *The relative income of deciles of ME society where the average of a decile is 0.1. The numbers calculated from Eq.(1). The left column is* n *and the right column is* $\varepsilon(n)$.

1.	0.289
2.	0.169
3.	0.120
4.	0.093
5.	0.076
6.	0.064
7.	0.056
8.	0.049
9.	0.044
10.	0.040

4. The Poverty

While Gini index and the ratio between deciles can be easily understood in terms of equilibrium society, poverty is harder to define. In the USA the poverty is defined as the inability to buy a certain amount of goods and services per unit time (i.e. a month). However, most countries define poverty as a relative quantity. In Europe a person is defined poor if his income is lower than 50% of the median income. The equilibrium network economy model cannot suggest the percentage of poor for the American absolute definition of poverty; however it can for the relative definition.

In Table 1 we see the equilibrium distribution of the wealth among the people according to their deciles. The median income which is given for a decile between the fifth and the sixth deciles is about 7 % of the total of the 10^{th} deciles. Half of this amount is 3.5%. Therefore, according to this definition, in country in equilibrium about 9% are poor. Indeed in the OECD countries the average percentage of poor is about this number. One should remember that the calculation of poverty as done by the countries' institutions is not so simple as the calculation is done per capita while the income is calculated per family, therefore the number of children might change the numbers. Nevertheless, the equilibrium figures are with very good agreement with OECD economies (Murtin, & d'Ercole, 2015).

5. The wealth of the rich as compared to the average

Economists usually express the income of upper deciles, percentile and tenth percentile in terms of the average income. To calculate the average income in the ME distribution we have to find $\bar{n}$ in which the sum of all the incomes below it is equal to the sum of the incomes above it, namely,

$$\sum_{n=1}^{\bar{n}} \ln(1+\frac{1}{n}) = \sum_{n=\bar{n}}^{N} \ln(1+\frac{1}{n}) \quad (3)$$

Which yields that; $2\ln(\bar{n}+1) = \ln(N+1)$ or

$$\bar{n} = \sqrt{N+1} - 1 \quad (4)$$

Using Eq. (4) we can calculate the ratio R of the income of the richest and the average income.

$$R_N = \ln 2 / \ln(1 + 1/\bar{n}) \quad (5)$$

It worth noting that R is a function of N. The higher is N, the higher the gap between the rich and the average. From Eqs. 5 and 4 we calculated table 2 which is the ratio between the upper fractions to the average. The left column is N and the right is R.

Table 2. *The ratio between the upper fractions to the average.The left column is N and the right is the wealth of richest fraction as compared to the average. As N increases the ratio increases.*

10	2
100	7
1000	22
10000	69
100000	219
1000000	693

From Eq. (5) we can calculate the equilibrium net income of the richest. For example, if the average yearly income of a person is 30K$, we see from table 2 that for deciles in which $N = 10$, the ratio between the upper decile and the average is 2, therefore the upper decile will make 60K$. Similarly, the upper percentile will make $7 \times 30 = 210$K$ and the upper tenth percentile annual income is $22 \times 30 =$ 660 K$.

6. CEO compensation

Eqs. (4) and (5) enable us to calculate the compensation of the CEO in terms of the average salary in his company and as a function of the number of employees N of the company. For example, Walmart has 2.2 million employees. In terms of an equilibrium company, its CEO should earn 1034 average salaries. Indeed, in reality Walmart's CEO makes a very similar number, namely 1028 average salaries (Link). In 23 companies of Fortune 100 the CEO compensation follows closely this formula. To mention few: Walmart, Macdonald's, Apple, Morgan Stanley, etc. Only 5 companies pay more than 2 times the equilibrium value, and in 5 companies the CEO makes less than 0.1 of this value. For example: W. Buffett salary is 0 in this scale. The average of the Fortune 100 companies is 0.87 as compared to 1 if all the companies would pay according to Planck Benford'slaw. Namely, on the average, for various reasons, there is a small tendency to pay a little less than the equilibrium salary for CEO's, the reasons probably are similar in their nature to that causing Mr. Buffett basically not to draw salary.

7. Pareto Law

Economists also calculate the distribution of wealth in term of the relative part of the total wealth held by the richest. The calculations in equilibrium society are done here for percentile using the equation $P = \ln 2/\ln 101$ for the first percentile, for the first ten percentiles $P = \ln 11/\ln 101$, and for the first 20%, $P = \ln 21/\ln 101$, of the wealth. This formula yields that the upper percentile has 15% of the total wealth, the upper decile has 52%, and the upper fifth has 66% of the wealth. In the OECD countries the average of the upper percentile has 18% of the wealth and the upper decile has 50% of the wealth (Murtin, & d'Ercole, 2015). 20% of the population in ME society has only about 66% of the wealth, a bit more justice for the poor than in the famous 80:20 Pareto law. According to this formalism 60% of the poorer population have 19.5% of the total wealth as compared to 13% in OECD.

8. Discussion

It is surprising that this oversimplified toy model yields such sound results. Yet, we have to point outthe limitations of this model. In this model there is only one bank and one country. In reality there are several banks and several countries trading between themselves. Moreover, the central bank takes no interest or any other fee for the loans. Yet it seems as if the single country inequality values are not affected by international trade or by the plurality of the banks or the charges of the bank. The second limitation is the differences between photons' energy and human wealth. With analogy to blackbody radiation in which all the photons in a given radiation modes have the same energy, the basic assumption of this model is that all the people in the same income rank earn exactly the same amount of money. The higher the ranknumber, the poorer the people (for large n the wealth ε is proportional to $1/n$ which is Zipf's law (Zipf, 1949; Gabarix, 1999). Therefore, when we divide the population to percentiles instead of deciles, we add more wealth scales of poor people that were not previously counted. For photons, the size of the blackbody determines the photons' minimum energy; similarly for people, the minimum money required to keep one alive determines the minimum wealth. This amount is lower than that of the formal definition of poverty in the OECD. This explains the differences of the ratio between the upper decile to the tenth decile, 9.6 in the OECD as compared to 7.25 of the present model, as some of the people that are poorer than the 10th decile of the model appears in the OECD statistics but not here. On the other hand, if we divide the people's wealth to percentiles instead of deciles, we count many poor people that are below the poverty that exists in the OECD. This limitation of "empty" percentiles of high rank number does not exist when calculating the CEO's compensation of companies in which the rank's number is low. The reason for it is that here we calculated the top salary in comparisonto the average salary which is substantially higher that the median salary. Generally, in the zone that $n \ll N$ the ratios of wealth will behave according to this model.

The same statistics was previously shown (Kafri, 2016) to be effective for voting. The distribution of the parliament seats among the 10 parties in Israel in the elections of 2015 is similar to that obtained by Eq.(1). In fact, the Gini index of inequality of seats among parties in the Israeli parliament, when is calculated according to Eq. (2) is 0.324. If so, one may ask whetherwe behave as a microcanonical ensemble after all. If we accept the assumption that the only physical law that causes irreversible changes in the universe is the second law, than the answer is that in equilibrium, maximum entropy distribution will be reached, and its probability should apply to economy which is a part of nature. As physicist Josiah Gibbs said (Kafri, & Kafri, 2013): "the whole is simpler than the sum of its parts".

9. Summary

In this note we calculate the thermodynamic equilibrium distribution of the wealth among people according to their income rank. We use a toy model economy of people randomly exchanging money between them selves. We make an analogy between this network economy and Planck's statistics in which the people/nodes are the photons, their energy is their wealth, and the social ranks are the radiation modes. We calculate for this distribution the indexes used by economists to describe the relative inequalities of income in countries and in companies. Namely, Gini Index, the ratio between highest income decile and the lowest income decile, relative poverty and the relative income compared to the average of the upper percentile and tenth percentile and the wealth held by the richest. We applied this formulation to calculate the executive compensation as a function of the number of employees and the average salary paid by the companies. The results fit well the inequities of wealth both in the OECD countries and in the Fortune 100 companies.

Acknowledgment

We thank Burke Brown for many comments and references of previous works and H. Kafri for many discussions.

References

Ball, P. (2004). The physical modeling of human social systems - A Review, *ComPlexUs*, 1, 190-206. doi. 10.1159/000082449

Dragulescu, A. & Yakovenko, V.M. (2000). Statistical Mechanics of Money, *European Physical Journal*, 17, 723-729. doi. 10.1007/s100510070114

Gabarix, X. (1999). Zipf's law for cities: An Explanation, *The Quarterly Journal of Economics*. 114(3), 739-767. doi. 10.1162/003355399556133

Kafri, O., & Kafri, H. (2013) *Entropy - God's dice game*. CreateSpace, pp. 154-157.

Kafri, O. (2014). Money, information and heat in networks dynamics, *Mathematical Finance Letters* 4, 1-11.

Kafri, O. (2016). A novel approach to probability, *Advances in Pure Mathematics*, 6, 201-211. doi. 10.4236/apm.2016.64017

Ksenzhek, O., & Petrova S. (2008) Inequality and economic efficiency of society through the prism of thermodynamics, Hungarian Electronic Journal Economics, Manuscript No. ECO-080111-A.

Murtin, F., & d'Ercole, M.M. (2015). Household wealth inequality across OECD countries: New OECD evidence, *OECD Statistic Brief*, No.21.

Planck, M. (1901) On the law of distribution of energy in the normal spectrum, *Annalen der Physik*, 4, 553-562.

Zipf, G.K. (1949). *Human Behavior and the Principle of Least-Effort*, Addison-Wesley.